Lecture 2

Important sets and subsets in topology

2.1 Euclidean spaces

2.1.1 The real line

As you know, $\mathbb{R}$ is the set of all real numbers. So $\mathbb{R}$ contains numbers such as 0, 1/3, -5, π, e, and $\sqrt{2}$. One can visualize $\mathbb{R}$ as a line. Indeed, it is sometimes referred to as "the real line."

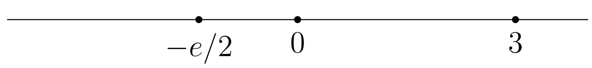

Figure 2.1: The real line $\mathbb{R}$, with some elements indicated.

2.1.2 The plane

$\mathbb{R}^2$ is the set consisting of all pairs (x_1, x_2) where both x_1 and x_2 are real numbers. In previous classes, you may have been used to writing an element of $\mathbb{R}^2$ as (x, y) instead. Well, letters are precious, so in our class, we will often write (x_1, x_2) instead.

Though we have only defined $\mathbb{R}^2$ as a set so far, as you know, it can be visualized as the usual "x-y plane." An element of $\mathbb{R}^2$ can be visualized as a point, or dot, inside the x-y plane. The *origin* is the point $(0, 0)$. Other points are as follows:

$$(1,1), \qquad (3,0), \qquad (3/4, 1/\pi), \qquad (\frac{-e}{2}, 0), \qquad (-3, \pi)$$

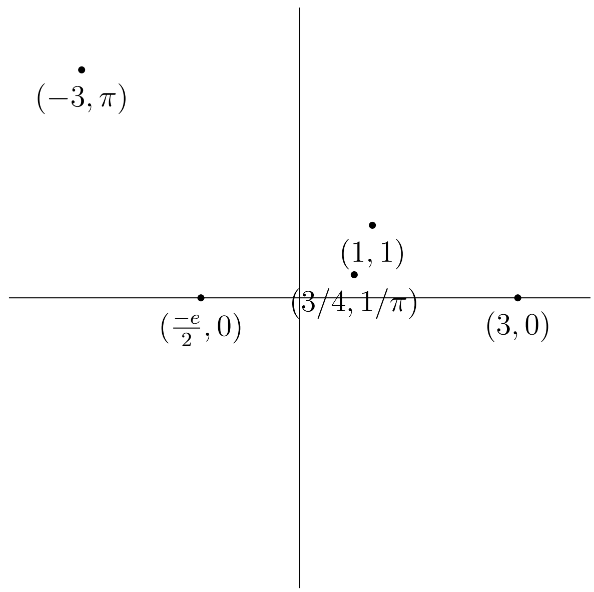

Figure 2.2: The plane, $\mathbb{R}^2$, with some elements labeled.

Note that $\mathbb{R}^2$ seems to contain a "copy" of $\mathbb{R}$, often called the x-axis, or in our class, the x_1-axis. It has a "vertical" copy of $\mathbb{R}$ as well, often called the y-axis, or in our class, the x_2-axis.

2.1.3 Three-dimensional Euclidean space

$\mathbb{R}^3$ is the set consisting of all triples (x_1, x_2, x_3) each x_i (for $i = 1, 2, 3$) is a real number. This set can also be visualized as "three-dimensional space," where the origin $(0, 0, 0)$ is placed at the center, and the numbers x_1, x_2, x_3 are the distances from the so-called "coordinate planes" of space. These distances uniquely determine a point "floating" in space.

Another way to think about the coordinates x_1, x_2, x_3 of a point are as instructions: Begin at the origin. Walk x_1 units along the x_1-axis, then x_2 units in the direction of the x_2-axis, then x_3 units in the direction of the x_3-axis.

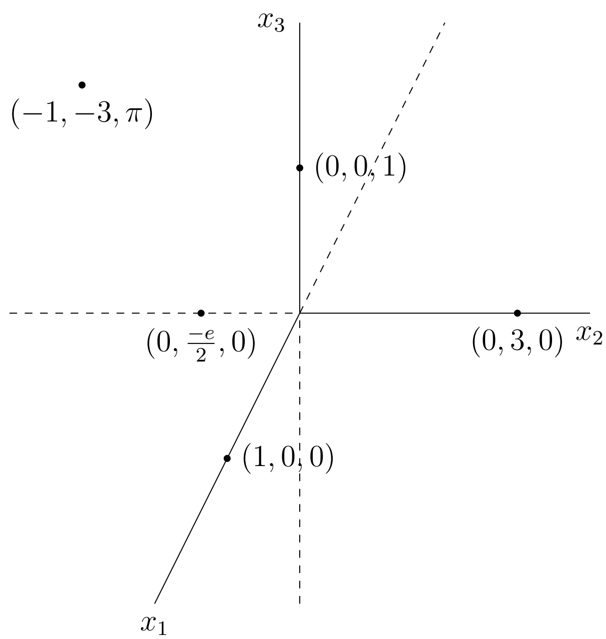

Figure 2.3: Euclidean space, $\mathbb{R}^3$, with some points labeled. Drawn also are the x_1- ,x_2-, and x_3-axis. The positive parts of these axes are drawn in solid, while the negative parts of the axes are drawn in dashes.

2.1.4 Higher-dimensional Euclidean spaces

More generally, for any integer $n \geq 0$, $\mathbb{R}^n$ denotes the set of all n-tuples, $(x_1, x_2, \ldots, x_n)$ where each x_i is a real number. For example, an element of $\mathbb{R}^4$ is a quadruple (x_1, x_2, x_3, x_4) of four real numbers. These sets are much harder to visualize, but are natural sets to consider.

By convention, $\mathbb{R}^0$ is declared to be a set with exactly one element. Informally, $\mathbb{R}^0$ is just "a point."

Note that there is a natural function p from $\mathbb{R}^4$ to $\mathbb{R}^3$ that "forgets" the last coordinate. That is, it sends

$$p : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3).$$

Thus, though subsets of $\mathbb{R}^4$ are hard to visualize (because $\mathbb{R}^4$ is hard to visualize) we often take a subset $A \subset \mathbb{R}^4$ and examine its image $p(A) \subset \mathbb{R}^3$ to begin to understand A.

2.2 Intervals

Intervals are very nice subsets of $\mathbb{R}$. If you choose any pair of real numbers a and b such that $a \leq b$, we can define intervals

$$[a, b], \qquad (a, b), \qquad (a, b], \qquad [a, b).$$

The first interval is a *closed* interval (it includes the endpoints a and b) while the second interval is an *open* interval (it does not include the endpoints). The last two intervals are neither open nor closed.

Of course, be warned that (a, b) here is the usual open interval inside $\mathbb{R}$; it does *not* symbolize a point in $\mathbb{R}^2$. The fact that these notations are "double-booked" is just a fact of history and life; we must live with it. It is often clear from context whether (a, b) refers to an interval or to a point of $\mathbb{R}^2$.

Of course, there are other intervals, such as:

$$(a, \infty), \qquad (-\infty, b), \qquad (-\infty, \infty), \qquad [a, \infty), \qquad (-\infty, b].$$

All these intervals are infinitely long. The first two intervals are open. The last two are closed.

Do you know whether the interval $(-\infty, \infty)$ (this interval is actually the entirety of $\mathbb{R}$) is considered open or closed (or both, or neither)?

2.3 Simplices

Definition 2.3.1. Fix an integer $n \geq 0$. We let the *standard n-simplex* be the subset

$$\Delta^n := \{(x_1, x_2, \ldots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i = 1 \text{ and for all } i,\ 0 \leq x_i.\} \subset \mathbb{R}^{n+1}.$$

Sometimes, we will just call Δ^n the *n-simplex*, dropping the word "standard."

The plural form of simplex is *simplices*.

Note that, by definition, the n-simplex is a subset of $\mathbb{R}^{n+1}$. So for example, the 2-simplex is a subset of $\mathbb{R}^3$.

Sometimes, we will re-index the coordinates and write an element of $\mathbb{R}^{n+1}$ as a tuple $(x_0, x_1, \ldots, x_n)$. (Note that the indexing here begins with 0, not 1.)

Exercise 2.3.2. (a) Draw the n-simplex, and how it sits inside $\mathbb{R}^{n+1}$, for $n = 0, 1, 2$.

(b) How many points does Δ^0 contain?

(c) (Optional.) Do you know what the 3-simplex looks like?

Believe it or not, the simplices turn out to be among the most important shapes in all of topology.

2.4 Spheres

Definition 2.4.1. Fix an integer $n \geq 0$. We let the *n-dimensional sphere* to be the subset

$$S^n := \{(x_1, x_2, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1.\} \subset \mathbb{R}^{n+1}.$$

Sometimes, we will call S^n the *n-sphere*, dropping the word "dimensional."

Exercise 2.4.2. (a) Draw the n-sphere, and how it sits inside $\mathbb{R}^{n+1}$, for $n = 0, 1, 2$.

(b) How many points does S^0 contain?

(c) Is there another name for the 1-sphere?

(d) (Optional.) Do you know what the 3-sphere looks like?

Spheres (of all dimensions) are also among the most important shapes in math.

2.5 Disks

Definition 2.5.1. Fix an integer $n \geq 0$. We let the *n-dimensional closed disk* to be the subset

$$D^n := \{(x_1, x_2, \dots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leq 1.\} \subset \mathbb{R}^n.$$

Sometimes, we will call D^n the *closed n-disk*, dropping the word "dimensional."

Exercise 2.5.2. (a) Draw the closed n-disk, and how it sits inside $\mathbb{R}^n$, for $n = 0, 1, 2, 3$.

(b) How many points does D^0 contain?

(c) Is there another name for the closed 1-disk?

(d) Is there some relationship between the "boundary" of the closed n-disk and the $(n-1)$-sphere?

Closed disks (of all dimensions) are also among the most important shapes in math.

2.6 Balls, open and closed

Fix a point $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$. Fix also a real number $r > 0$.

Definition 2.6.1. The *closed ball of radius r centered at y* is the set

$$\overline{\text{Ball}(y, r)} := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} (x_i - y_i)^2 \leq r^2\} \subset \mathbb{R}^n.$$

The *open ball of radius r centered at y* is the set

$$\text{Ball}(y, r) := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} (x_i - y_i)^2 < r^2\} \subset \mathbb{R}^n.$$

Don't be confused: y is not some other coordinate; y is another *point* in $\mathbb{R}^n$; that is, an element of $\mathbb{R}^n$. In past classes, you may have written $\vec{y}$, but we will be lazy and simply write an element of $\mathbb{R}^n$ as y, and denote its coordinates by $y_1, \ldots, y_n$.

Exercise 2.6.2. (a) Draw (in $\mathbb{R}^2$) the closed ball of radius 3 centered at $y = (3, 0)$.

(b) Is there a relationship between the closed ball of radius 1 centered at the origin of $\mathbb{R}^n$, and the closed n-disk?

(c) Draw (in $\mathbb{R}^2$) the open ball of radius 3 centered at $y = (3, 0)$.

(d) In $\mathbb{R}^n$, the open ball of radius 1 centered at the origin, and the closed ball of radius 1 centered at the origin, are different sets. How does the "difference" between these two sets relate to spheres?

2.7 Cubes

Definition 2.7.1. The *closed unit n-cube* is the set

$$\{(x_1, \ldots, x_n) \,|\, \text{for all } i,\ 0 \leq x_i \leq 1\} \subset \mathbb{R}^n.$$

Exercise 2.7.2. (a) Draw, for $n = 1, 2, 3$, the unit n-cube.