Lecture 4

Posets inside power sets, and intersections/unions

4.1 Every poset embeds into some power set

We've been exploring the notion of a partially ordered set, or poset for short. Posets are pedagogically useful because we can play with posets that are *finite* (like $[n]$, as opposed to $\mathbb{R}$) and at least they're somewhat intuitive. The easiest posets to think about are the $[n] = \{0 < 1 < \ldots < n\}$. And another natural poset is the power set $\mathcal{P}(A)$ of some set A.

In fact, *every* poset arises as a subposet of a power set.

Proposition 4.1.1. Let $(P, \leq)$ be a poset, and let $(\mathcal{P}(P), \subset)$ be the power set of P (with the usual "containment" partial order).

Then P is isomorphic, as a poset, to a subposet of $\mathcal{P}(P)$.

Exercise 4.1.2. Consider the function $j : P \to \mathcal{P}(P)$. It sends an element $q \in P$ to the set $j(q) = \{q' \mid q' \leq q\} \subset P$.

I claim that P is isomorphic (as a poset) to the subset $j(P)$—the image of j—endowed with the poset structure inherited from $\mathcal{P}(P)$.

Prove this, thereby proving Proposition 4.1.1.

Remark 4.1.3. Why is this Proposition useful? Well, you might be afraid that posets in general are allowed to look like crazy things. But the Proposition gives you some idea that the craziness can't be that bad. It says that nom matter what your poset P is, it arises by choosing some power set $\mathcal{P}(A)$, and then choosing some elements of the power set. If you like, you can think of power sets like cubes (using the drawing trick we saw last time), and the Proposition says that every poset arises by simply choosing some vertices of the cube.

Example 4.1.4. Consider the poset $[1] = \{0 < 1\}$. The power set has four elements, and can be drawn as

$$\begin{array}{ccc} \{0\} & \longrightarrow & \{0,1\} = [1] \\ \uparrow & & \uparrow \\ \emptyset & \longrightarrow & \{1\} \end{array}$$

The morphism j from the exercise is a function $j : [1] \to \mathcal{P}([1])$. It sends

$$0 \mapsto \{a \,|\, a \leq 0\} = \{0\}, \qquad 1 \mapsto \{a \,| a \leq 1\} = \{0,1\}.$$

So $[1]$ can be found inside $\mathcal{P}([1])$ as follows:

$$\begin{array}{ccc} \Large\{0\} & \longrightarrow & \Large\{0,1\} = [1] \\ \uparrow & & \uparrow \\ \scriptstyle\emptyset & \longrightarrow & \scriptstyle\{1\} \end{array}$$

(I've indicated the image of j using bigger font.)

Example 4.1.5. You should work out how $[2]$ sits inside the cube $\mathcal{P}([2])$ as the elements $\{0\}, \{0,1\}, \{0,1,2\}$.

4.2 Unions

I know you've gone over this in a previous class, but I want to be explicit about unions of sets. We'll talk about some very special kinds of unions, which we'll see in topology class all the time. It's not intuitive at first, but you'll get used to it with some practice.

4.2.1 $\mathcal{A}$-indexed collection of subsets of X

Let X be a set, and $\mathcal{P}(X)$ the power set. (As a reminder, $\mathcal{P}(X)$ is the collection of all subsets of X. It's a "bag of bags.") We'll let $\mathcal{A}$ be some other set, and choose a function

$$\mathcal{A} \to \mathcal{P}(X).$$

Concretely, for every element of $\mathcal{A}$, we specify a subset of X. Some call this a "$\mathcal{A}$-indexed collection of subsets of X." It's also a way of labeling subsets of $\mathcal{P}(X)$ (certain bags) with labels taken from the set $\mathcal{A}$. In this class, we will often denote elements of $\mathcal{A}$ by α, and we will denote the subset specified by α (via the above function) to by U_α. So the above function sends

$$\alpha \mapsto U_\alpha.$$

Some people will, instead of writing the function above, be lazy and write

$$\{U_\alpha\}_{\alpha \in \mathcal{A}}.$$

This is technically a less robust notation, because the collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ *loses* information if $\mathcal{A} \to \mathcal{P}(X)$ is not an injection. But you don't need to worry about this point, it won't make any difference to us.

Example 4.2.1. Let $X = \mathbb{R}$. If $\mathcal{A} = \{a, b, c\}$ is a set with three elements, then a function $\mathcal{A} \to \mathcal{P}(X)$ is simply a choice of three subsets of $\mathbb{R}$. These three subsets will be called U_a, U_b, U_c. And when we write $\alpha \in \mathcal{A}$, this just means that α could take on the values a, b, or c. In many of our examples, when $X = \mathbb{R}$, we'll often take each U_α to be some open interval.

Example 4.2.2. Let $X = \mathbb{R}$ again. We will let $\mathcal{A}$ be the set of all pairs (x, r) where $x \in \mathbb{R}$ and r is a positive real number. So $\mathcal{A}$ is a subset of $\mathbb{R}^2$. In terms of formulas,

$$\mathcal{A} := \{(x, r) \,|\, r > 0\} \subset \mathbb{R}^2.$$

I will now define a function

$$\mathcal{A} \to \mathcal{P}(X), \qquad (x, r) \mapsto (x - r, x + r)$$

sending an element $\alpha = (x, r)$ to the open interval centered at x and having width $2r$. So

$$U_\alpha = U_{x,r} = (x - r, x + r) \subset X.$$

In words, I am simply observing that for every pair of numbers x and r, we can define some open interval with center x and radius r.

Believe it or not, this notation is meant to be *convenient*. We'll often want to play with many subsets of X at once. And for our sanity, we'll want to be able to communicate which subsets we're talking about, i.e., give names to these subsets. The elements of $\mathcal{A}$ allow us to name the subsets. To each α is associated the subset U_α. You can think of α as the name, or label, given to U_α, if you like. And $\mathcal{A}$ is just a big set of possible names.

And often, we'll choose "names" that actually describe something about the set. For example, if I want to describe an interval $(x-r, x+r) \subset X = \mathbb{R}$ centered at x and with radius r, I might just name that interval "(x, r). " So for example, below we'll see notation like $U_{x,r} = (x-r, x+r) \subset \mathbb{R}$.

4.2.2 Unions of collections of subsets

Now suppose we have fixed a set $\mathcal{A}$, a set X, and a function

$$\mathcal{A} \to X, \qquad \alpha \mapsto U_\alpha.$$

Definition 4.2.3. The union

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset X$$

is the set of all elements of X that appear in at least one U_α. In set notation:

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha := \{x \in X \mid \text{there exists some } \alpha \text{ so that } x \in U_\alpha\}$$

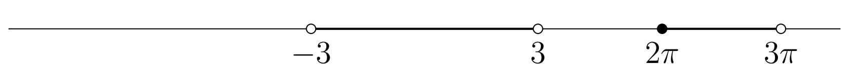
Example 4.2.4. Let $X = \mathbb{R}$. Let $\mathcal{A} = \{a, b, c\}$ and define

$$U_a = \{0\}, \qquad U_b = (-3, 3), \qquad U_c = [2\pi, 3\pi).$$

The the union is

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = U_a \cup U_b \cup U_c = \{0\} \bigcup (-3, 3) \bigcup [2\pi, 3\pi).$$

In terms of pictures, this union looks as follows:



Example 4.2.5. Let $X = \mathbb{R}$, and let $\mathcal{A} = \{(x,r) \,|\, x \in \mathbb{R}, r > 0\} \subset \mathbb{R}^2$. Define, for every $(x,r) \in \mathcal{A}$

$$U_{(x,r)} = (x-r, x+r) \subset X.$$

Then

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is the union of all the intervals $(x-r, x+r)$ for varying choices of $x \in \mathbb{R}$ and $r > 0$. Claim: This union is, in fact, the entirety of $X = \mathbb{R}$.

First, take a moment to try to convince yourself of this claim.

Now, how would we write a proof of this statement? Remember that to prove two sets are equal, the most efficient way is to show that each is a subset of the other. So we must prove two statements: $X \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ and $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset X$:

- Proof that $X \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$: Let $x \in X = \mathbb{R}$. Choose any $r > 0$. Then clearly $x \in (x-r, x+r)$. In particular, choosing $\alpha = (x,r)$, we have shown that $x \in U_\alpha$. Hence x is in the union.

- Proof that $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset X$: This is clear, because a union of subsets of X is still a subset of X. More pedantically, if $x \in U_\alpha$, then $U_\alpha \subset X \implies x \in X$. So every element in the union (which is necessarily an element of U_α for some α) is also an element of X.

4.3 Intersections

As usual let's fix a set X, a set $\mathcal{A}$, and a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$.

Definition 4.3.1. The intersection

$$\bigcap_{\alpha \in \mathcal{A}} U_\alpha$$

is the set of those x that appear in U_α for *every* $\alpha \in \mathcal{A}$. Put another way,

$$\bigcap_{\alpha \in \mathcal{A}} U_\alpha := \{x \in X \,|\, \forall \alpha \in \mathcal{A}, x \in U_{\mathcal{A}}\}.$$

Remark 4.3.2. Note that even if each U_α is not empty, it is (very!) possible that the intersection is empty.

Example 4.3.3. Let $X = \mathbb{R}$. Let $\mathcal{A} = \{a, b, c\}$ and define

$$U_a = \{0, 1\}, \qquad U_b = (-3, 3), \qquad U_c = [1, 5]$$

The the intersection is

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = U_a \cup U_b \cup U_c = \{0, 1\} \bigcup (-3, 3) \bigcup [1, 5].$$

The only real number inside all three of U_a, U_b, U_c is the real number 1. So the intersection is the one-element set $\{1\} \subset \mathbb{R}$.

Example 4.3.4. Let $X = \mathbb{R}$, and let $\mathcal{A} = \{(x, r) \,|\, x \in \mathbb{R}, r > 0\} \subset \mathbb{R}^2$. Define, for every $(x, r) \in \mathcal{A}$

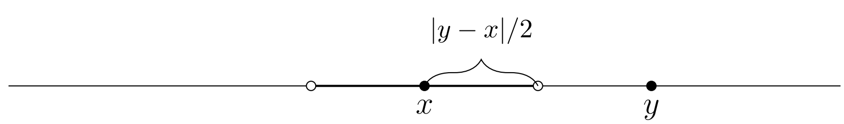
$$U_{(x,r)} = (x - r, x + r) \subset X.$$

Then

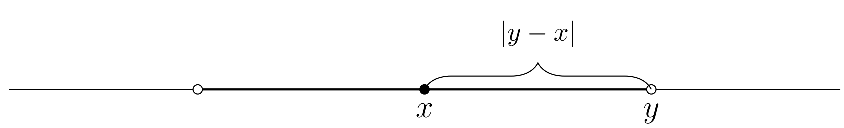
$$\bigcap_{\alpha \in \mathcal{A}} U_\alpha$$

is the intersection of all the intervals $(x-r, x+r)$ for varying choices of $x \in \mathbb{R}$ and $r > 0$. Claim: This intersection is the empty set.

To see this, I have to show that there is no real number y that is contained in *every* interval of the form $(x-r, x+r)$. In other words, given a real number y, I need to exhibit an interval $(x-r, x+r)$ in which y is not contained. This is fairly easy. For example, choose any real number $x \neq y$, and let $r = |x - y|$ be the distance between x and y. then y is not in the interval $(x-r, x+r)$. In fact, you could have also chosen r to equal $|x - y|/2$, too, if you wanted. You don't need to choose an "optimal" r, you just need to find one that works. Here's a picture if you chose r to be half the distance to y.



Here's the picture if you chose $r = |y - x|$. Note that y really is *not* inside $(x - r, x + r)$:



4.4 Get started on the homework quiz problems

If there is still time left in class, start working on the homework for this week, looking at the “quiz” multiple choice problems. Make sure you can draw the sets in question. Don’t just try to take this quiz on the fly; you’ll need to work out the answers ahead of time, else you probably won’t have enough time on the quiz.