Lecture 8

Subspaces

8.1 On some common phrases

8.1.1 Such that

I've noticed that many submissions are using the phrase "such that" incorrectly. This suggests that the phrase may also be misinterpreted, so let's be crystal clear.

"Such that" is completely synonymous with the following expressions:

1. "For which"
2. "Satisfying (the condition that)"
3. "Meeting the requirement that"
4. "For which it is guaranteed that"

For example, all the following sentences have the exact same content:

(a) Let T be a triangle such that all sides of T have equal length.

(b) Let T be a triangle for which all sides of T have equal length.

(c) Let T be a triangle for which all sides have equal length.

(d) Let T be a triangle meeting the requirement that all sides have equal length.

(e) Let T be a triangle meeting the requirement that all sides of T have equal length.

(f) Let T be a triangle for which it is guaranteed that all sides of T have equal length.

(g) Let T be a triangle satisfying the condition that all sides of T have equal length.

Which is to say, we are considering some equilateral triangle T.

As an example of the misuse of the phrase: Some of you have begun sentences with the words "such that." I am willing to bet a large sum of money that this sentence will end up not making sense!

8.1.2 "Bijective"

There is also a fundamental misconception people seem to have about the notions of surjection, injection, and bijection.

Remember that a function $f : X \to Y$ is called

- An injection if $f(x) = f(x')$ implies $x = x'$,
- A surjection if for any $y \in Y$, there is an $x \in X$ for which $f(x) = y$, and
- A bijection if f is both an injection and a surjection.

These definitions are about *functions*. In particular, functions are surjections/injections/bijections.

Yet people wrote things like "these two sets are injective." This does not make sense. It also does not make sense to say "these two sets are surjective."

For whatever reason, people seem to think that bijections are simply about a relationship between sets, and to forget about functions altogether. This is a grave misconception. In math—as you know from calculus—it is just as important (if not more important) to study *functions* as it is to study sets.

You should really think about bijections themselves as the information, and not some vague notion that two sets might "have the same size." The way in which you prove that two sets have the same size *is* through a bijection.

Likewise, while it is acceptable to say that two sets are "bijective," I may discourage you from doing so if it would prevent you would thinking that the sets are more important than the bijections themselves.

Moreover, people are still making some fundamental mistake: Claiming that "two sets have the same size, therefore there is a bijection between them." This kind of argument is almost always doomed to fail if the sets are infinite, because the only way to know if two infinite sets have "the same size" is precisely by exhibiting a bijection between them.

8.1.3 Let

I also want to comment on the phrase "Let X be a set." What does this phrase even mean?

Well, instead of answering that question, let me tell you what the phrase *does*.

Secretly, "Let X be a set" is just telling you what the symbol X will mean. It is telling you that X will be a symbol representing a set.

So you shouldn't think of "Let..." as being some deep, mysterious mathematical sentence. It's really just telling you that "if you see X later on, you know it's some set."

It's defining the *type* of the symbol X.

As another example, "Let $x \in X$" means now that lower case x will be an *element* of the set X. Again, it's telling you what *kind* of thing (i.e., what type) the symbol x represents.

Sometimes, you will see "Let $f : X \to Y$." Technically, this is not a complete English sentence, but it is a complete idea mathematically. It means that the symbol f will represent a function, and X will represent the domain of the function, while Y represents the codomain.

Note that I could have written "Let $\heartsuit$ be a set." The symbol doesn't matter; it doesn't need to even be a letter—though we tend to use letters of the Greek and Roman varieties.

So to summarize, "Let..." really conveys the following ideas:

- "I am setting the following notation"
- "The following symbol means..."

8.1.4 Definitions are abbreviations

I want you to know that most definitions are abbreviations. The definition of "democracy" tells us that we can use the word democracy as a very short (only four syllables!) way to describe a very wordy or complicated concept. It is the same in math. Any math definition (like "open") is just a short (two-syllable) way to express a precise idea that would take a long time to describe.

Over time and use, these abbreviations begin to take on intuitive meanings. I promise that you are developing these intuitions as we go on with this class, so be patient with yourself.

8.2 Subspaces: Universal property, definition, and examples

Now, onto the topic of today.

8.2.1 Motivation

Last time, we learned that the idea of "continuous function" has a huge generalization.[1] (It will be your homework, next week, to prove that, in fact, the notion of "continuous function" from last class is indeed compatible with the notion of "continuous function" from calculus class.) In fact, between any two topological spaces, and any function from one to the other, we can ask whether the function is continuous.

We know from calculus that continuous functions are special. They can be studied, because they have special properties. So with the blind faith that the math community is not introducing this new notion of continuity without merit, let's try to expand our ability to make new topological spaces.

For if we can make more topological spaces, we can apply the tools of continuous functions in more contexts.

[1] Generalization, in math, means that an idea that was only used in one context can now be used in many more contexts.

8.2.2 Review of some important functions

Note that in this section, no set is assumed to be a topological space, (so no function is assumed to be continuous[2]).

Definition 8.2.1. Let X be a set. Then the *identity function* of X is the function that sends any $x \in X$ to itself. We write

$$\mathrm{id} : X \to X$$

or, sometimes,

$$\mathrm{id}_X : X \to X.$$

Remark 8.2.2. The identity function has a very nice property—when you compose it with other functions, the other functions don't change.

More explicitly, let $f : X \to Y$ be some function. Then

$$\mathrm{id}_Y \circ f = f \qquad \text{and} \qquad f \circ \mathrm{id}_X = f.$$

I encourage you to verify this if you don't know why this is true.

So the word "identity function" is just a term for us to use. It's not a new idea. It's just a term. You're responsible for knowing that the term abbreviates.

Here is another important kind of function.

Definition 8.2.3. Let X be a set and let $A \subset X$. Then the *inclusion function* of A into X is the function sending any element $x \in X$ to itself. We denote the inclusion function by

$$i : A \to X, \qquad \text{or} \qquad i_A : A \to X.$$

Remark 8.2.4. There is a big difference between the meaning of "$A \subset X$" and the meaning contained in the function i. That A is a subset of X is a *property* of A, it's just something you can verify. But i is a very particular function. (Note that there are typically many, many functions from A to X.)

[2] Remember that it doesn't even make sense to ask whether a function is continuous if its domain and codomain aren't topological spaces.

Remark 8.2.5. Here is a very important property of the inclusion function. Suppose that there is another set W and a function $f : W \to X$. If the image of f is contained in A, then we can construct a new function $f' : W \to A$ such that

$$i_A \circ f' = f.$$

Confusingly, this f' is defined in the *exact same way* as f is, in terms of formulas. For all $w \in W$, we define f' to be the function with

$$f'(w) = f(w).$$

The big difference is that the codomain of f' is A, while the codomain of f is X itself. Then, indeed,

$$(i_A \circ f')(w) = i_A(f'(w)) = i_A(f(w)) = f(w).$$

Because this is true for all $w \in W$, we conclude that $i_A \circ f' = f$ as functions.

8.2.3 The universal property of subspaces

Let X be a topological space. (Remember, this means that X is given some topology $\mathcal{T}$.) And let $A \subset X$ be a sub*set*. Can we give A a topology that is somehow attuned to, or compatible with, the topology of X?

For example, can we guarantee that the inclusion function $i_A : A \to X$ is a continuous function?

The upshot is that yes, we can give A a topology such that i_A is continuous. In fact, we can do more:

Theorem 8.2.6 (Universal property of subspaces)**.** Let X be a topological space with topology $\mathcal{T}$, and fix any subset $A \subset X$. Then there exists a topology $\mathcal{T}_A$ on A satisfying the following properties:

1. The inclusion function $i_A : A \to X$ is continuous.

2. Moreover, let W be another topological space and $f : W \to X$ a continuous function for which $f(W) \subset A$. Then there exists a continuous[3] function $f' : W \to A$ such that

$$i_A \circ f' = f.$$

[3] Here, we are using $\mathcal{T}_A$ when discussing the continuity of f'.

3. Finally, f' is unique[4] among all continuous functions f' satisfying the equality $i_A \circ f' = f$.

The proof is given later in Section 8.3—just combine the three propositions in that Section.

8.2.4 Applications of universal property of subspaces

This Theorem is pretty amazing. You're not expected to process everything yet, so let me go through some examples.

Example 8.2.7. Let $X = \mathbb{R}^n$. Recall that last time, we gave X the *standard* topology. A set $U \subset X$ is in the topology $\mathcal{T}$ if and only if U is open (i.e., if it is a union of open balls).

We learned about many interesting *subsets* of X in week one. For example, we could choose $A \subset X$ to be

- The $(n-1)$-simplex, $\Delta^{n-1} \subset X$.
- The $(n-1)$-sphere, $S^{n-1} \subset X$.
- The n-cube.
- The closed n-dimensional unit disk.
- The open ball centered at some point $x \in \mathbb{R}^n$ with radius $r > 0$.

By the theorem above, we can consider *any* of these to be topological spaces.

Example 8.2.8. Moreover, thanks to the theorem, we can start producing continuous maps into these topological spaces.

For example, let's say that we want to find a continuous map from $\mathbb{R}$ into the sphere S^{n-1}. Then all we need to do is find a continuous map from $\mathbb{R}$ to $\mathbb{R}^n$ whose image lies inside the sphere. This is the power of Part 2 of the universal property of subspaces. (In the notation of the universal property—producing f allows you to conclude the existence of f'.)

[4] This means that if g is any other function satisfying $i_A \circ g = f$, then we can conclude that $g = f'$.

Example 8.2.9. Let $D^n \subset \mathbb{R}^n$ be the closed unit disk. (This is the set of all points with distance ≤ 1 from the origin of $\mathbb{R}^n$.)

Let $S^{n-1} \subset \mathbb{R}^n$ be the $(n-1)$-dimensional sphere. (This is the set of all points of distance exactly 1 from the origin of $\mathbb{R}^n$.)

By part 1 the theorem, we know that the inclusion map $i : S^{n-1} \to \mathbb{R}^n$ is continuous.

Moreover, by definition (of the sets S^{n-1} and D^n), the image of i is contained in D^n. So we may invoke Part 2 of the theorem, which tells us that there is a map

$$f' : S^{n-1} \to D^n$$

and this map is continuous. In fact, this is the *only* map (by Part 3) for which $i_{D^n} \circ f' = f$. In terms of notation (this is going to be confusing), we can write f' explicitly as follows: For every $x \in S^{n-1}$, we have that $f'(x) = x \in D^n$.

Isn't this great? We suddenly have hope of constructing continuous functions between various interesting spaces. In fact, we've shown that "including the sphere into the disk" is a continuous function (when we give the sphere S^{n-1} and the disk D^n the topology from the theorem above).

Remark 8.2.10. We just engaged in a very common exercise in math: Without *knowing* that a statement is true, we explored the consequences of the statement being true. We saw that the statement (the universal property of subspaces) seems very useful!

You secretly do this when you do a proof by contradiction. You explore how the universe would be if a statement were true; and if the universe comes upon a contradiction, you realize that the statement could not be true.

8.2.5 The subspace topology

The theorem has two parts: There *exists* a topology on $A \subset X$, and then there are the enumerated three properties that result from this topology.

This special property (resulting in the three properties of Theorem **??**) is as follows:

Definition 8.2.11 (Subspace topology)**.** Let X be a topological space. (We will call its topology $\mathcal{T}$.) Let $A \subset X$.

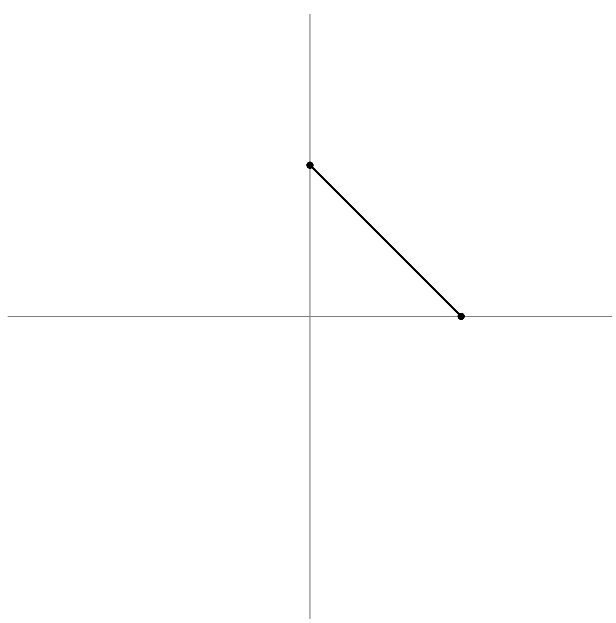
The *subspace topology* on A is the following collection of subsets of A:

$$\mathcal{T}_A := \{U \subset A \,|\, \text{there exists an open } V \subset X \text{ for which } U = V \cap A.\}$$

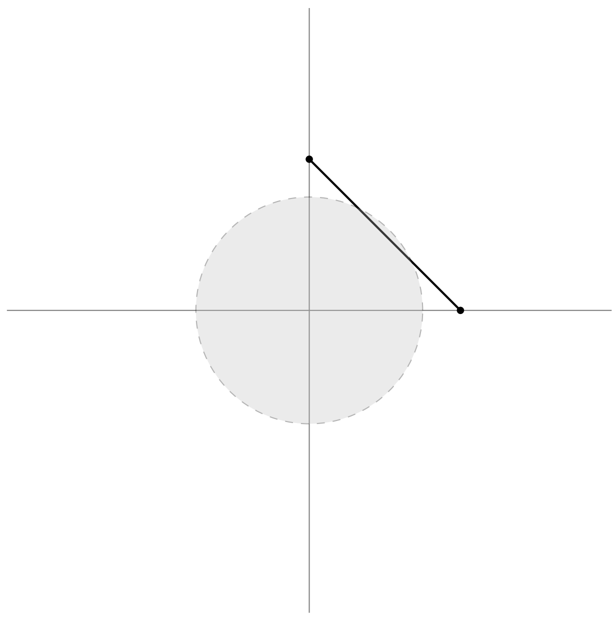
In words, the subspace topology $\mathcal{T}_A$ declares a subset of A to be open if and only if the subset arises as the intersection of A with an open subset of X.

8.2.6 Examples of open subsets of Δ^1 and D^2

Example 8.2.12. Let's look at one open subsets of Δ^1 look like. Remember, $\Delta^1 \subset \mathbb{R}^2$ is a "tilted" closed line segment in $\mathbb{R}^2$:

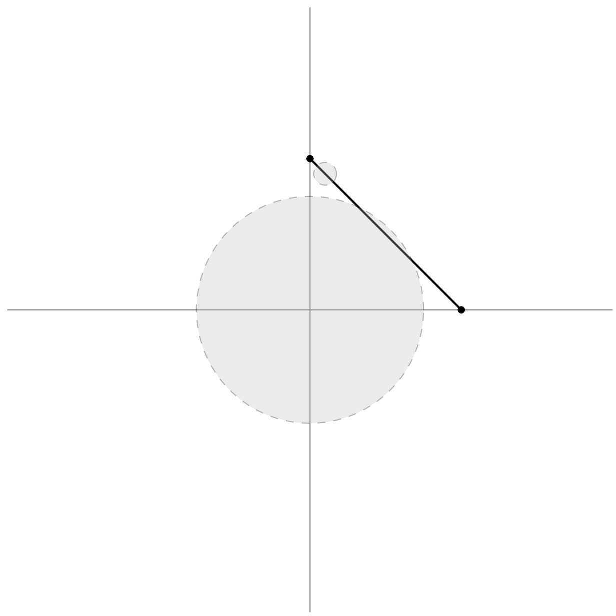


Taking an open ball in $\mathbb{R}^2$ centered at the origin, of radius, say, 1.5, let's intersect it with the 1-simplex:



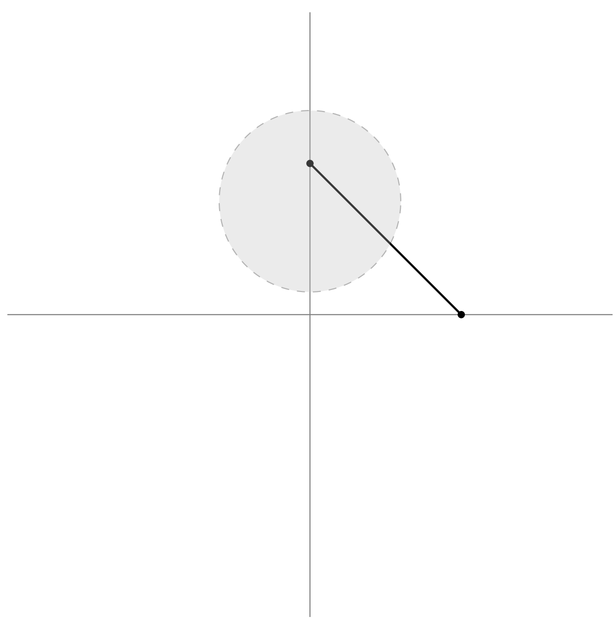
We see that an "open interval" inside of the 1-simplex is an open subset of the 1-simplex.

We can also take unions open balls in $\mathbb{R}^2$, and intersect the 1-simplex with such unions. Here is an example of a union of two open balls, intersected with the 1-simplex:



The intersection looks like two disjoint, open intervals inside Δ^1.

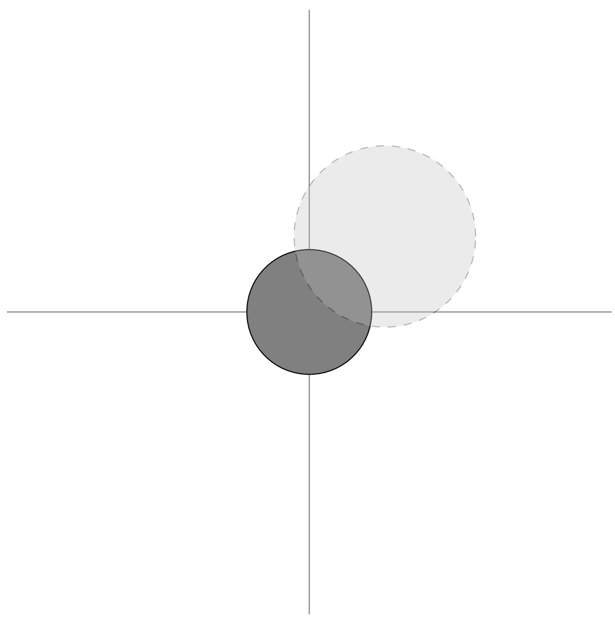
A more interesting example of open subset of Δ^1 is as follows:



Note that the intersection of this open ball with Δ^1 includes an "endpoint" of the 1-simplex. Indeed, the intersection looks like it is "closed" on one end (at the point $(0,1) \in \Delta^1 \subset \mathbb{R}^2$) but open at the other end. This is still an open subset of Δ^1!

What the lesson here is that—although it is still good to think of "open" as, in some cases, "lacking endpoints" or "lacking boundary points," this intuition can fail you when the shape you are in itself has boundary points or endpoints (like Δ^1).

Example 8.2.13. Now let $A = D^2$ be the closed unit disk. Then below is a picture of A, and an open ball inside $\mathbb{R}^2$;



The intersection is some open subset of D^2. Note that the intersection looks like a "lune," where on edge of the lune (laing on the boundary circle of D^2) is part of the subset, while the other edge of the lune (contained in D^2) is not; the two points at which these two points meet are also not in the open subset.

Remark 8.2.14. For any space, there are potentially a *lot* of open subsets. I would not dare to try to understand every open subset of $\mathbb{R}^n$, for example; it's just impossible.

But what math, and abstract reasoning, allows us to do is *still* prove something about the collection of open subsets without knowing what it is exactly. This is the power of abstract thought.

For example, you have no idea what $\sin(x)$ is for every value of x. But you can still say something about the function sin. You've simply gotten used to it, and feel happy about it, because you've worked with it some. It'll be the same for topologies.

8.3 Proving the universal property

There is a *lot* that needs to be verified. For example: Is $\mathcal{T}_A$ actually a topology? Then, does it really verify the three properties of the theorem?

Proposition 8.3.1. The subspace topology is a topology on A.

Proof. Remember that a collection of subsets of A is called a topology if the following three properties are satisfied.[5]

(i) The empty set and A are both in the collection.

(ii) The collection is closed under arbitrary unions.

(iii) The collection is closed under finite intersections.

First, we verify (i). Since $\mathcal{T}$ is a topology on X, we know that $\emptyset \in \mathcal{T}$. Moreover, $\emptyset \cap A = \emptyset$. Setting $U = \emptyset$ and $V = \emptyset$ we thus see that $U \in \mathcal{T}_A$ (by definition of $\mathcal{T}_A$).

Likewise, letting $V = X$ (which is in $\mathcal{T}$ by definition of topology on X) and noting that $A = V \cap A$, we see that $U = A$ is in $\mathcal{T}_A$ (by definition of $\mathcal{T}_A$).

Now we verify (ii). Suppose we have a collection of open subsets $\mathcal{A} \to \mathcal{T}_A, \alpha \mapsto U_\alpha$—so for each $\alpha \in \mathcal{A}$, each U_α is in the subspace topology $\mathcal{T}_A$. By definition of $\mathcal{T}_A$, then, we conclude that for all α, there is some $V_\alpha \in \mathcal{T}$ such that $V_\alpha \cap A = U_\alpha$. But we have that[6]

$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \left(\bigcup_{\alpha\in\mathcal{A}} V_\alpha\right) \cap A.$$

[5] See your notes from previous lecture.

[6] This is because $\left(\bigcup_{\alpha\in\mathcal{A}} V_\alpha\right) \cap A = \bigcup_{\alpha\in\mathcal{A}}(V_\alpha \cap A) = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$ where the last equality is how we chose the V_α. The first equality is a fact you should verify! You can also see Section 8.4.1.

Setting $V = \bigcup_{\alpha \in \mathcal{A}} V_\alpha$, we know that $V \in \mathcal{T}$ by definition of topology (on X). Hence by definition of $\mathcal{T}$, the equality above tells us that $\bigcup_{\alpha \ni \mathcal{A}} U_\alpha$ is in $\mathcal{T}_A$.

A similar proof verifies (iii). □

Proposition 8.3.2. Let X be a topological space with topology $\mathcal{T}$. Let $A \subset X$, endowed with the subspace topology. Then the inclusion map $i_A : A \to X$ is continuous.

Proof. By definition of continuity, we must prove that for every open subset $V \subset X$, the pre-image

$$i_A^{-1}(V) := \{a \in A \,|\, i_A(a) \in V\}$$

is in $\mathcal{T}_A$. By definition of i_A: For every $a \in A$, we know that $i_A(a) \in V \iff a \in V$. Hence $i_A^{-1}(V) = A \cap V$. But V is open, so $i_A^{-1}(V)$ is an element of cT_A by definition of $\mathcal{T}_A$. □

Proposition 8.3.3. Let $A \subset X$, and let $f : W \to X$ be a function such that $f(W) \subset A$. Define $f' : W \to A$ by $f'(w) = f(w)$. Then f' is continuous (with respect to the subspace topology of A) if f is continuous.

Proof. We must verify that the preimage of any open subset of A is open in W.

Let $U \subset A$ be open. Then $(f')^{-1}(U) = \{w \in W \mid f'(w) \in U\}$ by definition of preimage.

By definition of $\mathcal{T}_A$, we know there exists an open subset $V \subset X$ so that $U = V \cap A$. On the other hand, $f : W \to X$ is assumed continuous, so we know $f^{-1}(V)$ must be open in W. Note that

$$f^{-1}(V) = \{w \in W \,| f(w) \in V\}.$$

But because $f(W) \subset A$, we know that $f(w) \in V \iff f(w) \in V \cap A$. Thus

$$\begin{aligned} f^{-1}(V) &= \{w \in W \,| f(w) \in V\} \\ &= \{w \in W \,| f(w) \in V \cap A\} \\ &= \{w \in W \,| f(w) \in U\} \\ &= \{w \in W \,| f'(w) \in U\} \\ &= (f')^{-1}(U). \end{aligned} \tag{8.3.0.1}$$

We wanted to show that this last subset of W is open; since $f^{-1}(V)$ is open, we are done. □

Proposition 8.3.4. f' from above is the *unique* function from W to A which is continuous, and which satisfies $i_A \circ f' = f$.

Proof. In fact, f' is the unique function satisfying the equality $i_A \circ f' = f$. (This is just a matter of sets, and has nothing to do with spaces.)

To see this, if g is another function satisfying $i_A \circ g = f$, let us use the fact that i_A is injective to note that, for every $w \in W$,

$$i_A(f'(w)) = i_A(g(w)) \implies f'(w) = g(w).$$

Since $f'(w) = g(w)$ for every $w \in W$, f' and g are (by definition of function) the same function. □

8.4 Some comments to help

Note that all of our proofs today only involved intersections and unions and preimages. This is more or less because all our *definitions* involve only these notions. (How do you define continuity, and how do you define topology?)

Proofs that only require the kinds of ingredients that went into the definitions and statements are often regarded as "straightforward" in math. (Though, as you know from experience, these proofs can be anything but straightforward when one first gets used to the techniques.) They will be straightforward to you, too, if you get used to these techniques.

By the way, one reason that these proofs only require such tools is because we are only *given* such tools by the hypotheses. We are proving statements about *every* topological space, and we haven't seen whether every topological space even has a notion of distance (they don't! $\mathbb{R}^n$ is special!), so all we *can* use is things about unions, intersections, et cetera.

So, strangely enough, the foundations of topology just rest on basic facts about intersections, unions, and pre-images.

Next week, we'll see some results (like the Heine-Borel theorem) that begins to actually use $\mathbb{R}^n$ a little more.

8.4.1 Unions/intersection verification

In the proof of Proposition 8.3.1, I claimed

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \left(\bigcup_{\alpha \in \mathcal{A}} V_\alpha \right) \cap A.$$

where the V_α were defined to be (open) subsets of X for which $V_\alpha \cap A = U_\alpha$. Let's verify this.

To verify the equality of sets, we need to show $\subset$ and $\supset$.

Proof that $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset (\bigcup_{\alpha \in \mathcal{A}} V_\alpha) \cap A$. I will write this proof in a different format, in case it helps. You can feel free to write proofs in this format, too:

Suppose x is in $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

1. Then $x \in U_\alpha$ for some $\alpha \in \mathcal{A}$ (by definition of union).
2. Thus $x \in V_\alpha \cap A$ (because V_α is a set chosen to satisfy $V_\alpha \cap A = U_\alpha$).
3. Hence $x \in \bigcup_{\alpha \in \mathcal{A}} (V_\alpha \cap A)$ (by definition of union).
4. Now I claim $\bigcup_{\alpha \in \mathcal{A}} (V_\alpha \cap A) = (\bigcup_{\alpha \in \mathcal{A}} V_\alpha) \cap A$.
   (a) $\subset$: If $y \in \bigcup_{\alpha \in \mathcal{A}} (V_\alpha \cap A)$, then for some α, $y \in V_\alpha \cap A$. In particular, y is in V_α for some α, hence inside $\bigcup_{\alpha \in \mathcal{A}} V_\alpha$. And because $y \in V_\alpha \cap A$, y is an element of A, too.

   We have shown that y is an element of both A and of $\bigcup_{\alpha \in cA} V_\alpha$.

   (b) $\supset$: Assume that y is in both $\bigcup_{\alpha \in \mathcal{A}} V_\alpha$ and A. Then for some $\alpha \in \mathcal{A}$, we know that $y \in V_\alpha$. In particular, $y \in V_\alpha \cap A$. Thus $y \in \bigcup_{\alpha \in \mathcal{A}} (V_\alpha \cap A)$.
5. We have shown that $x \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ implies $x \in (\bigcup_{\alpha \in \mathcal{A}} V_\alpha) \cap A$.

□

Proof that $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \supset (\bigcup_{\alpha \in \mathcal{A}} V_\alpha) \cap A$. Suppose x is in $(\bigcup_{\alpha \in \mathcal{A}} V_\alpha) \cap A$. Note that by line 4 above, we may thus conclude that $x \in \bigcup_{\alpha \in \mathcal{A}} (V_\alpha \cap A)$.

1. Then $x \in V_\alpha \cap A$ for some $\alpha \in \mathcal{A}$ (by definition of union).
2. Thus $x \in U_\alpha$ for some $\alpha \in \mathcal{A}$ (by definition of V_α).
3. Thus $x \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ (by definition of union).

□