Lecture 10

Compactness, II. Heine-Borel theorem.

Last time we learned about covers, open covers, and subcovers. After seeing examples, we culminated with the definition of compactness.

We saw that finite spaces are compact. That posets with minimal elements are compact, and (without proof) that $[0,1] \subset \mathbb{R}$ is compact when equipped with the subspace topology.

We saw that $\mathbb{R}^n$ is not compact.

10.1 A little review

Proposition 10.1.1. If X and Y are homeomorphic, then X is compact if and only if Y is compact.

A proof of this is a riff on Proposition 9.4.4 from the previous lecture notes, so is omitted here.

Proposition 10.1.2. Let a, b be real numbers such that $a < b$. Then the interval $[a, b]$ is homeomorphic to the interval $[0, 1]$. (When both are given the subspace topology inherited from $\mathbb{R}$.)

The proof of the above proposition was contained in Corollary 9.6.2 from last time's lecture notes.

Combining these two propositions, we conclude that any interval of the form $[a, b]$ is compact.

10.2 You can pull back open covers

Also, I mentioned in class, but not in the notes, the following fact. You may use this freely:

Proposition 10.2.1. Let $f : X \to Y$ be a continuous function, and let $\mathcal{V}$ be an open cover of Y. Define

$$\mathcal{U} := \{U \subset X \,|\, U = f^{-1}(V) \text{ for some } V \in \mathcal{V}.$$

Then $\mathcal{U}$ is an open cover of X.

The math lingo for this proposition is that open covers "pull back." We call $\mathcal{U}$ the *pull-back* of $\mathcal{V}$. I'll give a proof here, though you should practice proving it on your own:

Proof. To show $\mathcal{U}$ is an open cover, we must show that it is a cover, and that every element of $\mathcal{U}$ is open.

$\mathcal{U}$ **is a cover**. We must show that every $x \in X$ is an element of some $U \in \mathcal{U}$. Given x, consider the point $f(x) \in Y$. We know that $\mathcal{V}$ is a cover of Y, so there exists some $V \in \mathcal{V}$ for which $f(x) \in V$. In particular, $x \in f^{-1}(V)$ by definition of preimage.

On the other hand, $U = f^{-1}(V)$ is an element of $\mathcal{U}$ by definition of $\mathcal{U}$. So $x \in U$, and we are finished with this claim.

$\mathcal{U}$ **is an open cover**. We must show that every $U \in \mathcal{U}$ is open. Well, $U \in \mathcal{U}$ if and only if $U = f^{-1}(V)$ for some $V \in \mathcal{V}$ (by definition of $\mathcal{U}$). And any $V \in \mathcal{V}$ is known to be open (because $\mathcal{V}$ is assumed to be an *open* cover). Because f is continuous, the pre-image of any o pen subset is open—in particular, $U = f^{-1}(V)$ is open. □

A special example of this is when A is a subset of X, and $i_A : A \to X$ is the inclusion function. Then any open cover of the big space X "pulls back" to an open cover of A. In this case, $i_A^{-1}(V)$ has a particular simple interpretation, as

$$i_A^{-1}(V) = A \cap V.$$

10.3 The Heine-Borel Theorem

Let's keep building up our examples of compact spaces. There's a theorem called the Heine-Borel theorem that actually tells us *exactly* all the subsets of $\mathbb{R}^n$ that are compact (with respect to the subspace topology).

To state the theorem succinctly, we'll want to learn another term.

Definition 10.3.1. Let $A \subset \mathbb{R}^n$. We say that A is *bounded* if there exists some positive real number $r > 0$ such that

$$A \subset \text{Ball}(0, r).$$

In other words, there is some big number r so that *every* point of A is at most r units away from A.

Equivalently, A is called bounded if there is some $r > 0$ such that, for every $a \in A$, we have that $\text{dist}(0, a)$—the distance from the origin to a—is less than r.

Remark 10.3.2 (Bounded sequences)**.** In analysis, you may have heard about "bounded" sequences. This is a similar use of the word bounded. A sequence $a_1, a_2, \ldots$ is called bounded if there is some $r > 0$ such that for every i, $a_i \in \text{Ball}(0, r)$.

In real analysis, you also hear about sequences being bounded above or bounded below. We won't use these notions as much in this course. Regardless, I'll tell you what these mean. A sequence $a_1, a_2, \ldots$ is *bounded above* if there is some real number t such that, for every i, $a_i < t$.

Likewise, a sequence $a_1, a_2, \ldots$ is *bounded below* if there is some real number s such that, for every i, $a_i > s$.

Given a sequence $a_1, a_2, \ldots$, the following are equivalent:

(i) The sequence is bounded.

(ii) The sequence is bounded above and bounded below.

I promise you can prove the equivalence of these two statements on your own. But now, we will ignore the notion of boundedness for sequence, and concentrate on the notion of boundedness in Definition 10.3.1.

Now, it turns out we can completely describe which subsets of $\mathbb{R}^n$ are compact:

Theorem 10.3.3 (Heine-Borel theorem)**.** Let A be a subset of $\mathbb{R}^n$.

Then A is compact if and only if A is both closed and bounded.

Remark 10.3.4. Note that being closed and bounded are properties of A as a *subset* of $\mathbb{R}^n$. When we endow this subset with the subspace topology inherited from $\mathbb{R}^n$, we may make sense of what it means for A to be compact. And that is what we mean.

So, put in a more wordy fashion, the Heine-Borel theorem says: A, with the subspace topology, is compact if and only if A is closed and bounded as a subset of $\mathbb{R}^n$.

We won't prove the Heine-Borel theorem today. But you may use it freely from now on.

To make good use of the Heine-Borel theorem, we'll want tools to decide when a subset of $\mathbb{R}^n$ is closed, and when it is bounded.

The homework due on Tuesday will have you determine whether certain subsets of $\mathbb{R}^n$ are closed, are bounded, both, or neither.

Because it's usually straightforward to decide whether a subset of $\mathbb{R}^n$ is bounded or not (you just need to determine whether elements in that subset can be arbitrarily far away from the origin) we'll focus on a study of closed subsets of $\mathbb{R}^n$.

10.4 Closed subsets

Throughout this section, we're going to assume we have placed the standard topology on $\mathbb{R}^n$.

10.4.1 Some easy closed subsets of $\mathbb{R}^n$

By definition, a subset of $\mathbb{R}^n$ is closed if and only if its complement is open.

Example 10.4.1. Let $A = \{x \in \mathbb{R}^n \mid \text{dist}(0, x) \geq 5\}$. That is, this is the set of points that are distance 5 or greater from the origin.

Then A is closed. This is because the complement is an open ball of radius 5 centered at the origin—and open balls are certainly open.

Example 10.4.2. Let $A = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$. This is otherwise known as the x_2-axis. Then the complement of A is open, as we saw in a previous lecture note. So A is closed.

By their very definition, closed subsets often require us to demonstrate something about complements of sets. So most proofs involving closed subsets will involve some ingredient of complements; so you will want to be fluent with computing complements.

10.4.2 Some closed subsets of $\mathbb{R}$

And we can also find some closed subsets of $\mathbb{R}$ straightforwardly:

Proposition 10.4.3. The following are all closed subsets of $\mathbb{R}$:

1. A set consisting of a single point.
2. For any real number a, the intervals $[a, \infty)$ and $(-\infty, a]$.

Proof. Choose a point $x \in \mathbb{R}^n$. We must show that $A = \{x\}$ is closed. There are two ways to see this. Any sequence $a_1, a_2, \ldots$,in A is a "constant" sequence, and hence converges to x (which is in A). Thus A satisfies the "convergent sequence" criterion of closedness you proved in homework.

Another way to see this: A^C is equal to $\mathbb{R} \setminus \{x\}$. This can be written as a union of open intervals of finite radius, so is open.

As for $[a, \infty)$, let $\mathcal{A}$ denote the collection of all pairs $(x, r) \in \mathbb{R} \times \mathbb{R}_0$ so that the intersection of the open interval $(x - r, x + r)$ and $[a, \infty)$ is empty. Letting $U_{(x,r)}$ be the open ball of radius r about x—also known as the open interval $(x - r, x + r)$—we see that the complement of $[a, \infty)$ is the union

$$\bigcup_{(x,r) \in \mathcal{A}} \mathrm{Ball}(x, r).$$

So $[a, \infty)$ has open complement. The proof for $(-\infty, a]$ is nearly identical. □

10.4.3 Ways to make new closed subsets

Now that we have some simpleton collections of closed subsets (of $\mathbb{R}$ and of $\mathbb{R}^n$) let's see if we can make some more sophisticated ones.

As I mentioned before, because the definition of closedness involves complements, so will many proofs. The above properties are a straightforward consequence of *de Morgan's laws*, which are the most useful tools for understanding unions and intersections of complements. We will assume these without proof:

Proposition 10.4.4 (de Morgan's laws)**.** The complement of a union is the intersection of the complements. And the union of complements is the complement of the intersection.

In symbols, for any collection of sets $\{B_\alpha\}_{\alpha\in\mathcal{A}}$, we have that

$$\left(\bigcup_{\alpha\in\mathcal{A}} B_\alpha\right)^C = \bigcap_{\alpha\in\mathcal{A}} B_\alpha^C, \qquad \bigcup_{\alpha\in\mathcal{A}} B_\alpha^C = \left(\bigcap_{\alpha\in\mathcal{A}} B_\alpha\right)^C$$

(ii) and (iii) below helps us make new closed subsets out of old ones:

Proposition 10.4.5. Let X be a topological space with topology $\mathcal{T}$. Let $\mathcal{K}$ be the collection of closed subsets of X (so that $B \in \mathcal{K} \iff B^C \in \mathcal{T}$). Then the following hold:

(i) $\emptyset, X \in \mathcal{K}$.

(ii) $\mathcal{K}$ is closed under finite unions. That is, if $B_1, \ldots, B_n$ are closed subsets of X, then the union $B_1 \bigcup B_2 \ldots \bigcup B_n$ is also a closed subset of X.

(iii) $\mathcal{K}$ is closed under arbitrary intersections. That is, if we have an arbitrary collection $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ of closed subsets of X, then the intersection $\bigcap_{\alpha\in\mathcal{A}} B_\alpha$ is also a closed subset of X.

(ii), in plain English, says the finite union of closed sets is closed. (iii) says that the intersection of closed sets is closed. You see the similarity with the axioms of a topology, except that the finiteness condition is swapped.

Proof of Proposition 10.4.5. (i). Because $\emptyset$ is open (by definition of topology), and $\emptyset^C = X$, X is closed. Likewise, because X is open (by definition of topology) and $X^C = \emptyset$, we see that $\emptyset$ is closed. We've shown that X and $\emptyset$ are elements of $\mathcal{K}$.

(ii). Let $B_1, \ldots, B_n$ be a finite collection of closed subsets. We must show that $B_1 \bigcup \ldots \bigcup B_n$ is closed. In other words, we must show that $(B_1 \bigcup \ldots \bigcup B_n)^C$ is open. By de Morgan's laws, we see

$$\left(B_1 \bigcup \ldots \bigcup B_n\right)^C = B_1^C \bigcap \ldots \bigcap B_n^C.$$

Since each B_i is closed by assumption, we know that each B_i^C is open. Thus the above is an intersection of finitely many open subsets of X. Such an intersection is known to be open by definition of topology.

(iii). Finally, given an arbitrary collection $\{B_\alpha\}_{\alpha \in \mathcal{A}}$ of closed sets, we must show that the intersection is closed. That is, we must show that $\left(\bigcap_{\alpha \in \mathcal{A}} B_\alpha\right)^C$ is open. Well, by de Morgan's laws,

$$\left(\bigcap_{\alpha \in \mathcal{A}} B_\alpha\right)^C = \bigcup_{\alpha \in \mathcal{A}} B_\alpha^C.$$

This last expression is a union of open subsets of X. By definition of topology, such a union is open. ☐

Example 10.4.6. We saw earlier that $[a, \infty) \subset \mathbb{R}$ is closed. Likewise, we know that $(-\infty, b]$ is closed. If we assume $a > b$, we see that the closed interval $[a, b] \subset \mathbb{R}$ is closed (because intersections of closed sets are closed).

Given some real numbers $a_1 < a_2 < \ldots < a_{2n}$, the union

$$[a_1, a_2] \bigcup [a_3, a_4] \bigcup \ldots \bigcup [a_{2n-1}, a_{2n}]$$

is closed, because a finite union of closed subsets is closed.

Likewise, if $A \subset \mathbb{R}$ is a finite subset, then A is closed. This is because A can be written as a finite union of singleton sets (i.e., sets with exactly one element in them), and we saw that singleton sets are closed.

10.4.4 Making closed sets using preimages

The following is another method of creating closed subsets:

Proposition 10.4.7. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Then if $B \subset Y$ is closed, then $f^{-1}(B)$ is closed.

Proof. If $B \subset Y$ is closed, then B^C is open in Y. Because f is continuous, $f^{-1}(B^C)$ is open in X. Moreover,

$$\begin{aligned} f^{-1}(B^C) &= \{x \in X \mid f(x) \in B^C\} \\ &= \{x \in X \mid f(x) \notin B\} \\ &= \{x \in X \mid x \notin f^{-1}(B)\} \\ &= (f^{-1}(B))^C. \end{aligned}$$

That is, the preimage of a complement is the complement of a preimage. So $f^{-1}(B)$ has an open complement. By definition, this means $f^{-1}(B)$ is closed. ☐

Remark 10.4.8. In fact, a function $f : X \to Y$ is continuous if and only if the preimage of any closed subset of Y is closed. So this is an equivalent definition of continuity.

In other words, once you start having a wealth of closed subsets of some space Y, and a wealth of continuous functions to Y, then we can start discovering many closed subsets of X.

Theorem 10.4.9. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}$, both with the standard topology. Then any function that is a finite sum or product of the "standard" functions from calculus—polynomials in each coordinate, sin or cosine, et cetera—is continuous.

The proof of the above theorem isn't so bad, but I'll leave it as an extra credit assignment for future weeks. It's the kind of fact that many students would rather assume, so they can move on with their lives. So I won't dwell on it.

You may use the above theorem freely from now on.

Example 10.4.10. The following are all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$:

(a) $(x_1, x_2, \ldots, x_n) \mapsto x_1$.

(b) $(x_1, x_2, \ldots, x_n) \mapsto x_2$.

(c) For any i between 1 and n, the projection map $(x_1, x_2, \ldots, x_n) \mapsto x_i$.

(d) $(x_1, x_2, \ldots, x_n) \mapsto x_1^2 + x_2^2 + \ldots + x_n^2$.

(e) $(x_1, x_2, \ldots, x_n) \mapsto \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. (This is known as the "distance from the origin" function.)

(f) More generally, for any $y \in \mathbb{R}^n$, the function $x \mapsto \text{dist}(x, y)$ is continuous. (This is the "distance from y" function.)

(g) $f(x_1, \ldots, x_6) = \sin(x_1)x^2 + \cos(x_4)e^{x_5} - \pi x^6$ is another example.

Given this wealth of continuous functions, we can now exhibit all kinds of closed subsets of $\mathbb{R}^n$.

Exercise 10.4.11. (a) $S^{n-1} \subset \mathbb{R}^n$ is closed. This is because S^{n-1} is the preimage of the one-element set $\{1\} \subset \mathbb{R}$ (which is closed) with respect to the function $x \mapsto \operatorname{dist}(0, x)$ (which is continuous). Note that this sphere is also bounded, so S^{n-1} is compact when given the subspace topology inherited from $\mathbb{R}^n$.

(b) The unit disk $D^n \subset \mathbb{R}^n$ is closed. This is because D^n is the preimage of the interval $[0, 1]$ (which is closed in $\mathbb{R}$) with respect to the function $x \mapsto \operatorname{dist}(0, x)$ (which is continuous). Since D^n is bounded, D^n is compact by the Heine-Borel theorem.

(c) Fix an integer i between 1 and n. The "ith upper half space" $\{x \in \mathbb{R}^n \mid x_i \geq 0\}$ is closed. This is because this is the preimage of the interval $[0, \infty)$ (which is closed in $\mathbb{R}$) under the projection map $x \mapsto x_i$ (which is continuous). This is not bounded, so the upper half space is closed and not compact.

(d) The "non-negative octant"

$$\{x \in \mathbb{R}^n \mid \text{For all } i,\, x_i \geq 0$$

is closed. This is because the octant is the intersection of the upper half-spaces from the previous example (and an intersection of closed sets is closed.) This set is also not bounded.

(e) The hyperplane

$$\{(x_0, \ldots, x_n) \mid \sum_{i=0}^{n} x_i = 1\} \subset \mathbb{R}^{n+1}$$

is closed, because this set is the preimage of the singleton set $\{1\}$ (which is closed in $\mathbb{R}$) under the function $(x_0, \ldots, x_n) \mapsto x_0 + x_1 + \ldots + x_n$ (which is continuous). This set is not bounded.

(f) The n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is closed. This is because the n-simplex can be expressed as the intersection of the hyperplane from the previous example with the positive octant. This set is bounded, so the n-simplex (given the subspace topology) is compact by the Heine-Borel theorem.

(g) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the graph of f, which is defined to be the subset of $\mathbb{R}^2$ defined by

$$\{(x_1, x_2) \mid f(x_1) - x_2 = 0\}$$

is a closed subset of $\mathbb{R}^2$. This is because this is the preimage of $\{0\}$ (which is a closed subset of $\mathbb{R}$) under the function $\mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1) - x_2$. This is a continuous map because it is obtained by combining continuous functions together. (The notion of "combining" is left vague for the moment.) This set is not bounded—for example, there are points on the graph of f with arbitrarily large x_1 coordinate.

(h) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, fix a real number a, and consider the set

$$\{x \mid f(x) \geq a\}.$$

This is a closed subset of $\mathbb{R}^n$. This is because the set is a pre-image of the set $[a, \infty)$ (which is closed in $\mathbb{R}$) under the map $x \mapsto f(x) - a$.

You can take finite unions and arbitrary intersections of the above examples to produce many, many closed subsets of $\mathbb{R}^n$.

10.5 Useful facts about compactness

We haven't seen yet why compactness is useful. Here is a first sign:

Proposition 10.5.1. Let X be compact.[1] Fix a continuous function $f : X \to Y$. Then the image of f—endowed with the subspace topology inherited from $\mathcal{Y}$—is compact.

In words, the image of a compact space is compact (so long as the image is taken with respect to a continuous function).

Proof. We must show that any open cover of $f(X)$ admits a finite subcover.

By definition of subspace topology, a subset $V \subset f(X)$ is open if and only if $V = W \cap f(X)$ for some open subset $W \subset Y$. So, fixing an open cover $\mathcal{V}$ of $f(X)$, we know that for every $V \in \mathcal{V}$, we may find some $W \subset Y$ for which $V = W \cap f(X)$.

[1] Note that at this point, we may infer that X is a topological space; this is because "compact" is an adjective that only makes sense for topological spaces.

Let $f' : X \to f(X)$ denote the continuous function guaranteed by the universal property of the subspace topology.[2] Because f' is continuous, we may pull back $\mathcal{V}$ along f' and obtain an open cover of X. (This is Proposition 10.2.1.)

Concretely, this pull-back cover is

$$\mathcal{U} = \{U \subset X \,|\, U = (f')^{-1}(V) \text{ for some } V \in \mathcal{V}.$$

(As defined in Proposition 10.2.1.)

Now we use the fact that X is compact. $\mathcal{U}$ is an open cover, so by compactness of X, $\mathcal{U}$ admits a finite subcover. Let us denote the elements of this finite subcover by

$$U_1, \ldots, U_n.$$

Well, by definition, for each i, there is some $V_i \in \mathcal{V}$ such that $U_i = f^{-1}(V_i)$. So consider the collection

$$V_1, \ldots, V_n.$$

Note that each V_i is a subset of $f(X)$.

I claim this collection $\{V_1, \ldots, V_n\}$ is a cover of $f(X)$. To see this, we must show that every element of $f(X)$ is contained in some V_i. Well, let $y \in f(X)$. By definition, y is in the image of f, so there exists some $x \in X$ for which $f(x) = y$. Because the $U_1, \ldots, U_n$ are a cover of X, we conclude that $x \in U_i$ for some i. By definition, this U_i is the preimage of V_i, so we conclude that $f(x) \in V_i$. Recalling that we chose x so that $f(x) = y$, we conclude that the y we began with is an element of V_i. This shows that $\{V_1, \ldots, V_n\}$ forms a cover of $f(X)$.

So we have exhibited a finite subcover $\{V_1, \ldots, V_n\}$ of $\mathcal{V}$. □

Remark 10.5.2. Thus, compactness is one of the few properties that "push forward" under a continuous map. Usually, preimages behave well under continuity by definition. We've already seen examples where preimages of compact spaces need not be compact, but here we see that images of compact spaces are always compact.

In the above proof, we have put together some of the wonderful ingredients we've learned so far—the universal property of subspaces, and that open covers pull back, for example.

Here is a corollary of the above fact:

[2]Remember, in terms of formulas, $f'(x) = f(x)$. The meat of f' is the ability to change the codomain of f.

Corollary 10.5.3. Let X be a compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function. Then $f(X)$ is compact.

Now, let me state that it is *incredibly common* to study functions from a space into $\mathbb{R}$. For example, if X is the set consisting of points on the surface of the earth, $f : X \to \mathbb{R}$ could be a function sending a point x to the height of x above/below sea level. Or it could send x to the temperature at x, or the pressure, et cetera.

As you well know, there are points of "highest elevation" on earth. It also turns out that, if you assume the temperature function is continuous, there will (at any given moment time) be a point of highest and lowest temperature. This is a consequence of the fact that we can say what all compact subsets of $\mathbb{R}$ must look like! It turns out that every compact subset of $\mathbb{R}$ contains a maximal and minimal element. We will get to that later.

Here is another useful fact.

Proposition 10.5.4. Let X be a compact topological space, and let $A \subset X$ be a closed subset. Then A (with the subspace topology inherited from X) is compact.

In other words, closed subsets of compact spaces are compact.

Proof. We must show that every open cover of A admits a finite subcover.

Choose an open cover $\mathcal{U}$ of A. By definition of subspace topology, for every $U \in \mathcal{U}$ there is some $V \in \mathcal{T}_X$ such that $U = V \cap A$.[3] Choose one V for every $U \in \mathcal{U}$, and let $\mathcal{V}$ denote the collection of these chosen V. (So $\mathcal{V}$ is in bijection with $\mathcal{U}$.)

Now consider the collection $\mathcal{W} = \mathcal{V} \bigcup \{A^C\}$. That is, $\mathcal{W}$ consists of the sets V, and of another set called A^C. Note that A^C is open because A is closed; so every member[4] of $\mathcal{W}$ is open.

Moreover, I claim that $\mathcal{W}$ is a cover of X. Indeed, a point of x is either in A or not. If x is not in A, then $x \in A^C$, which is a member of $\mathcal{W}$. If $x \in A$, then there is some $U \in \mathcal{U}$ for which $x \in U$. In particular, x is in the V we chose to correspond to U.

This shows that $\mathcal{W}$ is an open cover.

[3] As usual, $\mathcal{T}_X$ stands for the topology on X.

[4] Member is a synonym for element. So a member of a set is the same thing as an element of a set.

By compactness of X, $\mathcal{W}$ admits a finite subcover. But if $\mathcal{W}$ is an open cover of X, its pull-back to A is an open cover of A. Moreover, we see that $i_A^{-1}(V_i) = V_i \cap A$ is precisely equal to some $U_i \in \mathcal{U}$. So this pullback is a finite subcover of $\mathcal{U}$, proving that A is compact. $\square$