Lecture 11

Compactness, III. Extreme value theorem.

11.1 John the aspiring ballerino

Here's a story.

John loved ballet. He would go watch ballet twice a week, over 90 minutes each time. He loved the movements, he loved the music, he loved everything about ballet and watched with a passion. He longed to be a ballerino.

He signed up for an audition for the local ballet troupe. On the day of his audition, he got up on stage. He was asked if he was familiar with Swan Lake. He said yes, he'd seen it over and over. He had watched YouTube videos analyzing dancers, had heard dancers speak about their experiences performing Swan Lake, he had seen it all.

John was asked to perform any sequence from it.

And of course, he could not. He had never danced on his own. He had never taken lessons, or had a teacher; but worst, he had never spent the hours necessary to practice, to watch himself in a mirror, to dance.

Watching lectures will not teach you math. You have to try to dance yourself, and to learn how to improve what you seen in the mirror. When you take an exam, you will not be assessed for the time you spent as an audience. You will be assessed for the time you spent alone practicing.

11.2 Three results today

Today we'll cover three results.

Proposition 11.2.1. Let $A \subset \mathbb{R}$ be compact.[1] Then there exists a maximal element in A.

More precisely, there exists $a \in A$ such that for all $a' \in A$, $a \geq a'$.[2]

Proof. By the Heine-Borel theorem, A is both closed and bounded.

Because A is bounded, there exists some $r > 0$ such that A is contained in the interval $(-r, r) \subset \mathbb{R}$. In particular, there is some real number b for which $a' \in A \implies a' \leq b$. Let us call such a number b an *upper bound* for A. (Note that there are infinitely many upper bounds for A.) By the least upper bound property of the real line[3], the set of upper bounds of A has a minimal element called b_0. In other words, b_0 is the smallest real number satisfying the upper bound property.

I claim that b_0 is an element of A. (This would prove the proposition.) To see this, for every integer $n \geq 1$, we simply choose an element $a_n \in A$ such that $\text{dist}(a_n, b_0) \leq 1/n$. This is possible because b_0 is the *least* upper bound.

Then the sequence $\{a_n\}_{n\geq 1}$ converges to b_0 (Given any $\epsilon > 0$, choose N large enough so that $1/N < \epsilon$, and we see that $\text{dist}(a_n, b_0) < \epsilon$ for all $n \geq N$).

But A is closed, so $b_0 \in A$. □

Proposition 11.2.2. Let X be compact, and $f : X \to Y$ a continuous function. Then the image of f (given the subspace topology inherited from Y) is compact.

Proof. We must prove that $f(X)$ is compact. Instead of the function $f : X \to Y$, consider the function $f' : X \to f(X)$ which sends any $x \in X$ to $f(x) \in f(X)$. The function f' is continuous by the universal property of the subspace topology (for $f(X)$).

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[1] By the Heine-Borel theorem, this means A is a closed and bounded subset of $\mathbb{R}$. Also as usual, when we say that A is compact, we are really endowing A with the subspace topology inherited from the standard topology on $\mathbb{R}$. This is important, because a set may admit many different topologies.

[2] Here we are using the usual order on $\mathbb{R}$—whether two numbers are less than or equal to each other.

[3] This is a property of the real line we won't go over in this class; you'll see it, or have learned about it, in analysis.

Let $\mathcal{V}$ be any open cover of $f(X)$. Then because f' is continuous, and by the construction of pullback open covers,

$$\mathcal{U} := \{U \subset X \,|\, U = (f')^{-1}(V) \text{ for some } V \in \mathcal{V}\}$$

is an open cover for X. Because X is compact, we may choose a finite subcover $\{U_1, \ldots, U_n\}$. For each $i = 1, \ldots, n$, choose $V \in \mathcal{V}$ to be an open subset for which $U_i = (f')^{-1}(V_i)$.

I now claim that the collection $\{V_1, \ldots, V_n\}$ is a subcover of $\mathcal{V}$. It suffices to show that $V_1 \bigcup \ldots \bigcup V_n = f(X)$. To see this, choose $y \in f(X)$. Then by definition of image, there is some $x \in X$ for which $f(x) = y$. Because $\{U_1, \ldots, U_n\}$ is an open cover, there is some i for which $x \in U_i$. In other words, there is some i for which $x \in (f')^{-1}(V_i)$, meaning $f'(x) \in V_i$. But $f'(x) = f(x) = y$, so we conclude $y \in V_i$. This shows that the union $V_1 \bigcup \ldots \bigcup V_n$ contains $f(X)$; because this union is a priori a subset of $f(X)$, we see that the union equals $f(X)$.

This completes the proof. □

Combining the above results gives us:

Theorem 11.2.3 (Extreme Value Theorem)**.** Let X be compact, and let $f : X \to \mathbb{R}$ be a continuous function. Then f attains a maximal value.

That is, there exists some $x \in X$ such that, for every $x' \in X$, we have that $f(x) \geq f(x')$.

Proof. The image $f(X) \subset \mathbb{R}$ is compact by Proposition 11.2.2. Thus it has a maximal value by Proposition 11.2.1. □

Remark 11.2.4. The condition that X be compact is necessary. For example, let $X = \mathbb{R}$ and let $f : X \to \mathbb{R}$ be the identity function, so $f(x) = x$. This attains no maximal value.

For the rest of today, you will choose either Proposition 11.2.2 or the Extreme Value Theorem, and you will try over and over until you can write the proof on your own.

This is similar to a dancer practicing until they can nail down a move, or to any other athlete/musician/artist honing their craft.