Lecture 13

Quotient spaces

Suppose that you are given a topological space $(X, \mathcal{T})$. Remember, this means that you are given a set X, and a topology $\mathcal{T}$—which is a way to declare certain subsets of X as "open." (The word "open" does not inherently have any meaning; it is $\mathcal{T}$ that contains the meat.)

It turns out we can construct a *new* topological space by "gluing" some points of X together, or by "identifying" them (meaning that you declare some points to "become equal").

This new space is called a *quotient* topological space. We'll learn today how to create these.

Today we're going to be working mostly in groups. You're supposed to get practice reading a mathematical statement on your own, then trying to find examples to help your understanding.

13.1 An example

Here is an intuitive example: If you take a shoelace, and you fuse the two plastic ends[1] together, you get a loop.

Mathematically, if you model the shoelace by the closed interval $X = [0, 1]$, the process of fusing the ends together can be modeled by creating a new set by demanding that the points $0 \in X$ and $1 \in X$ "become the same."

Today we'll see how to give $X/\sim$ a topology; that is, we won't just have a quotient *set*, but we will have a *quotient topological space*, which we will often call a *quotient space* for short.

[1] The plastic caps at the ends of a shoelace are called "aglets."

13.1.1 The quotient topology

Definition 13.1.1. Let $(X, \mathcal{T})$ be a topological space and fix an equivalence relation $\sim$ on X. The *quotient topology* on the quotient set $X/\sim$ is defined as follows:

A subset $U \subset X/\sim$ is declared open if and only if $q^{-1}(U)$ is open in X.

Here, $q : X \to X/\sim$ is the quotient map from last time.

Theorem 13.1.2 (Universal property of quotient spaces)**.** The quotient topological space $X/\sim$, and the quotient map $q : X \to X/\sim$, satisfy the following properties:

1. q is continuous.
2. Suppose we are given a topological space Y and a continuous map $f : X \to Y$, satisfying the property that $x \sim x' \implies f(x) = f(x')$. Then there exists a map $f' : X/\sim \to Y$ such that
   (a) f' is continuous, and
   (b) $f' \circ q = f$.
3. Moreover, f' is the *unique* map from $X/\sim$ to Y satisfying the two above properties.

Remark 13.1.3 (What are universal properties good for?)**.** Universal properties are supposed to allow you to be *lazy* (in a good way). A less judgmental word would be *efficient*. Usually, it takes a lot of work to define a function and prove that it is continuous. It takes even more work if you construct a new topological space and don't have much of a feel for this new space.

Well, $X/\sim$ is a new space, so you might not have a feel for it. But it's *easy* to construct continuous maps whose domains are $X/\sim$.

Why? If you understand X, then you often know whether a function $f : X \to Y$ is continuous. (For example, if X and Y are Euclidean spaces, you have lots of examples from calculus.) What the universal property says is that you just need to *check one condition* to make new functions out of $X/\sim$: Test whether $f(x) = f(x')$ each time x is related to x'.

If the function f passes this test, then you *automatically* are guaranteed a function called f' which has domain $X/\sim$ and codomain Y. This $X/\sim$ is the mysterious new space, yet you have a concrete tool for constructing functions out of it—just find functions f that pass the test.

Remark 13.1.4. This was also the utility of the universal property of subspace topologies. Both S^1 and S^2 are beautiful shapes and spaces, but as topological spaces, they may seem abstract and it may seem hard to construct continuous functions into them. Well, the universal property of subspace topologies makes things easy. If you want to make a continuous map to S^2, all you need to do is construct a continuous function to $\mathbb{R}^3$, and check that the image of your function is inside S^2.

13.2 Exercises

Exercise 13.2.1. Write down the equivalence relation on $X = [0,1]$ that declares 0 and 1 to be equivalent (without making any other identifications).

Make sure you can do this using $\sim$ notation, and using a subset $R \subset X \times X$.

Exercise 13.2.2. Let X be a topological space and $\sim$ an equivalence relation. Show that the quotient map $q : X \to X/\sim$ is continuous when $X/\sim$ is given the quotient topology.

Exercise 13.2.3. Let X be a topological space and let $R = \Delta$ be the *diagonal* equivalence relation. Show that the projection map $q : X \to X/\sim$ is a homeomorphism.

Exercise 13.2.4. Consider the function $j : \mathbb{R} \to \mathbb{R}^2$ given by

$$j(t) = (\cos(t), \sin(t)).$$

You may take for granted that j is continuous.

(a) Consider the interval $[0, 2\pi] \subset \mathbb{R}$. Show that the function

$$j' : [0, 2\pi] \to \mathbb{R}^2, \qquad t \mapsto (\cos(t), \sin(t))$$

is continuous. (Hint: Use that $i_{[0,2\pi]}$ is continuous, and that compositions of continuous functions are continuous.)

(b) Define an equivalence relation on $X = [0, 2\pi]$ be declaring that

$$t \sim t' \iff \begin{cases} t = t' \\ t = 0 \text{ and } t' = 2\pi \\ t' = 0 \text{ and } t = 2\pi \end{cases}$$

Prove that the function

$$j'' : X/\sim \to \mathbb{R}^2, \qquad [t] \mapsto (\cos(t), \sin(t))$$

is continuous. (Hint: Universal property of quotient spaces.)

(c) Prove that the function

$$j''' : X/\sim \to S^1 \qquad [t] \mapsto (\cos(t), \sin(t))$$

is continuous. (Hint: Universal property of subspaces.)

(d) Prove that j''' is a bijection.

In a later class, we will see that j''' is a homeomorphism.

Note that, *not once* did you have to know about open sets or the definition of continuity in the above exercises. The universal properties make things automatic.

Exercise 13.2.5. Give $\mathbb{R}$ the standard topology. Define an equivalence relation on $\mathbb{R}$ as follows:

$$t \sim t' \iff \text{For some non-zero } x, \text{ we have that } tx = t'.$$

(a) Verify that this is an equivalence relation.

(b) How many elements does $\mathbb{R}/\sim$ have?

(c) Write down every open subset of $\mathbb{R}/\sim$.

(d) Is $\mathbb{R}/\sim$ compact?

Exercise 13.2.6. Let $A \subset \mathbb{R}^2$ be the subset

$$\{(x_1, x_2) \,|\, x_1, x_2 \in [0, 1]\}.$$

(A is a square region.) We endow A with the subspace topology. We let

$$\partial A := \{(x_1, x_2) \in A \,|\, x_1 = 0 \text{ or } x_1 = 1 \text{ or } x_2 = 0 \text{ or } x_2 = 1.$$

(∂A is the "boundary square" of A.)

Define an equivalence relation on A as follows:

$$x \sim x' \iff \begin{cases} \text{both } x \text{ and } x' \text{ are in } \partial A & \text{or} \\ x = x'. \end{cases}$$

(a) Prove that the above is an equivalence relation.

(b) Do you have any guesses on what well-known space $A/\sim$ is homeomorphic to?

Exercise 13.2.7. Let $A \subset \mathbb{R}^2$ bed the subset

$$A = [0, 2\pi] \times [0, \pi] = \{(x_1, x_2) \,|\, x_1 \in [0, 2\pi] \& x_2 \in [0, \pi]\}.$$

Consider the function

$$f : A \to \mathbb{R}^2, \qquad (x_1, x_2) \mapsto (\cos(x_2)\sin(x_1), sin(x_2)\sin(x_1), \cos(x_1)).$$

Let $\sim$ be the equivalence relation on A given by

$$x \sim x' \iff f(x) = f(x').$$

Do you have any guesses as to what well-known shape $A/\sim$ is homeomorphic to?

Exercise 13.2.8. Choose a subset $A \subset X \times X$. Show that there exists a *smallest* equivalence relation R that contains A.

(Hint: Show that the intersection of equivalence relations is again an equivalence relation; then take the intersection of all equivalence relations that contain A. By the way, how do you that there exists at least one equivalence relation that contains A?)

We sometimes say that R is the equivalence relation *generated* by A.