Lecture 15

Product spaces and their universal property

15.1 Direct products (Cartesian products)

Let X and Y be two sets. Recall that the *direct product*, or the *Cartesian product* of X and Y is denoted

$$X \times Y$$

and is defined to be the set consisting of pairs (x, y) where $x \in X$ and $y \in Y$. As an example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and a point of $\mathbb{R}^2$ is precisely an (ordered) pair of real numbers.

One thing I would like to emphasize is that there are functions called *projection functions*, or *projection maps*, defined as follows:

$$p_X : X \times Y \to X, \qquad (x, y) \mapsto x$$

and

$$p_Y : X \times Y \to Y, \qquad (x, y) \mapsto y.$$

Here is a subtle fact about products of sets that you may not have learned about before: How do you define functions to $X \times Y$?

Proposition 15.1.1. Let W be a set. To give a function $f : W \to X \times Y$ is the same thing as giving a function $W \to X$ and a function $W \to Y$.

More precisely, there exists a bijection

$$\{f : W \to X \times Y\} \cong \{(W \to X, W \to Y)\}$$

between the set of functions from W to $X \times Y$, and the set of pairs of functions from W (to X and to Y).

Here is how the proof goes. Given a function $f : W \to X \times Y$, we can post-compose with the projection maps. This gives a pair of functions

$$p_X \circ f : W \to X, \qquad p_Y \circ f : W \to Y.$$

On the other hand, suppose you are given a pair of functions $g_X : W \to X$ and $g_Y : W \to Y$. This defines a function g as follows:

$$g : W \to X \times Y, \qquad g(w) = (g_X(w), g_Y(w)).$$

I claim that the assignments

$$\psi : f \mapsto (p_X \circ f, p_Y \circ f), \qquad \phi : (g_X, g_Y) \mapsto g$$

are mutually inverse to each other. Indeed,

$$(\phi \circ \psi)(f)(w) = ((p_X \circ f)(w), (p_Y \circ f)(w)) = f(w)$$

and

$$(\psi \circ \phi)(g_X, g_Y) = (p_X \circ g, p_Y \circ g) = (g_X, g_Y).$$

Example 15.1.2. A function from a set W to $\mathbb{R}^2$ is completely determined by a pair of functions from W to $\mathbb{R}$.

15.2 Product spaces

Now let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. It turns out we can endow $X \times Y$ with a topology called the product topology. We'll define it in a moment, but first, the universal property:

Theorem 15.2.1 (Universal property of the product topology)**.** Let X and Y be topological spaces. Then there exists a topology on $X \times Y$ satisfying the following properties:

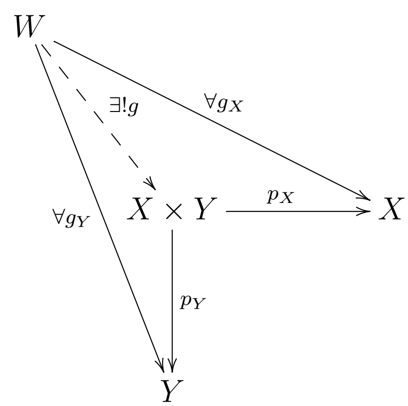
(i) Each projection map $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ is continuous.

(ii) For any topological space W, and for any pair of continuous functions $g_X : W \to X, g_Y : W \to Y$, there exists a continuous function $g : W \to X \times Y$ such that

$$g_X = p_X \circ g, \qquad g_Y = p_Y \circ g.$$

(iii) Moreover, g is unique among such functions. That is, if there is some other continuos function $h : W \to X \times Y$ for which $g_X = p_X \circ h$ and $g_Y = p_Y \circ h$, then $h = g$.

In terms of diagrams,



Again, universal properties are supposed to help you. For example, you might be afraid that $X \times Y$ is complicated. But it is *easy* to understand continuous functions to $X \times Y$. You just need to understand continuous functions to X, and continuous functions to Y. Then you understand all continuous functions to $X \times Y$.

So what is this magical topology?

Definition 15.2.2. Let X and Y be topological spaces. Then the *product topology* on $X \times Y$ is defined as follows. We declare a subset $U \subset X \times Y$ to be open if and only if U can be written as a union of sets of the form $V_X \times V_Y$, where $V_X \in \mathfrak{T}_X$ and $V_Y \in \mathfrak{T}_Y$.

Theorem 15.2.3. The product topology satisfies the properties of the theorem above.

Proof. (i) p_X is continuous, as for any open subset $U_X \subset X$, we can prove that $p_X^{-1}(U_X) = U_X \times Y$. Because $U_X \in \mathfrak{T}_X$ and $Y \in \mathfrak{T}_Y$, this set is open in the product topology of $X \times Y$. Likewise, p_Y is continuous.

(ii) Given $g_X : W \to X$ and $g_Y : W \to Y$, define $g(w) = (g_X(w), g_Y(w))$. Then it follows that $p_X \circ g = g_X$ and $p_Y \circ g = g_Y$. We must now prove that g is continuous. To see this, let $U \subset X \times Y$ be open. Then

$$U = \bigcup_{\alpha \in \mathcal{A}} V_\alpha \times V'_\alpha$$

where for all α, we know that $V_\alpha \in \mathcal{T}_X$ and $V'_\alpha \in \mathcal{T}_Y$. Thus

$$g^{-1}(U) = g^{-1}\left(\bigcup_{\alpha \in \mathcal{A}} V_\alpha \times V'_\alpha\right). = \bigcup_{\alpha \in \mathcal{A}} g^{-1}\left(V_\alpha \times V'_\alpha\right).$$

Now we note that, for all $\alpha \in \mathcal{A}$,

$$g^{-1}(V_\alpha \cap V'_\alpha) = g_X^{-1}(V_\alpha) \cap g_Y^{-1}(V'_\alpha).$$

This is a finite intersection of open subsets of W (because g_X and g_Y are continuous) hence for all α, the set $g^{-1}(V_\alpha \cap V'_\alpha)$ is open in W. We thus see $g^{-1}(U)$ is a union of of open subsets of W, hence is open.

This proves g is continuous.

(iii) If h is another such function, then for all w, we see that $p_X(h(w)) = p_X(g(w))$ and $p_Y(h(w)) = p_Y(g(w))$. It follows that $h(w) = g(w)$ for all w, hence $h = g$. □

In homework, you will prove that the standard topology on $\mathbb{R}^2$ agrees with the product topology on $\mathbb{R}^2$.

15.3 Direct products of infinitely many sets

Now let $\mathcal{A}$ be an arbitrary set. And for each $\alpha \in \mathcal{A}$, choose a set X_α.

Warning 15.3.1. This is like choosing a function from $\mathcal{A}$ to the "collection of sets." However, a famous paradox (Russell's paradox) proves that there is no such thing as the set of all sets. Strangely, it is intuitively healthy but logically unhealthy to think of the choice above as a function to the set of all sets.

Then the *direct product* of the X_α is the set of all collections $(x_\alpha)_{\alpha \in \mathcal{A}}$ for which $x_\alpha \in X_\alpha$ for all α. The direct product is denoted

$$\prod_{\alpha \in \mathcal{A}} X_\alpha.$$

In words, an element of $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is the same thing as a choice of element x_α in each X_α.

Example 15.3.2. Let $\mathcal{A} = \{1, 2\}$ be a two-element set, and choose two sets X_1 and X_2. Then there is a natural bijection

$$\prod_{\alpha \in \mathcal{A}} X_\alpha \cong X_1 \times X_2.$$

This is because an element of the right hand side is the choice of $x_1 \in X_1$ and $x_2 \in X_2$. Such a choice encodes exactly an ordered pair $(x_1, x_2) \in X_1 \times X_2$.

Example 15.3.3. Now let $\mathcal{A}$ be the set

$$\mathbb{Z}_{\geq 1},$$

the set of all positive integers, and choose a set X_i for every $i \in \mathbb{Z}_{\geq 1}$. Then

$$\prod_{i \in \mathbb{Z}_{\geq q}} X_i$$

is the collection of all sequences $(x_1, x_2, x_3, \ldots)$ where for each $i > 0$, x_i is an element of X_i.

In the special case that each X_i happens to be the set $\mathbb{R}$, then an element of $\prod_{i \in \mathbb{Z}_{\geq 0}} \mathbb{R}$ is the same thing as a sequence in $\mathbb{R}$, in the sense of calculus or analysis.

Example 15.3.4. Let $\mathcal{A}$ be the set of all particles in the universe, and for each particle $\alpha \in \mathcal{A}$, let X_α be the set of possible states that the the particle can be in. Then an element of the direct product

$$\prod_{\alpha \in \mathcal{A}} X_\alpha$$

is the data of choosing a state for every particle in the universe. The idea of "state" is ill-defined here, but you get the idea.

The important point here is that you can take direct products of possibly *infinitely many* sets (if $\mathcal{A}$ is infinite)—not just two.

Just as before, there are projection maps $p_\alpha : \prod_{\alpha \in \mathcal{A}} X_\alpha \to X_\alpha$ sending an element $(x_\alpha)_{\alpha \in \mathcal{A}}$ of the domain to x_α (you forget all the components except the α component). And as before, for any set W, to give a function $f : W \to \prod X_\alpha$ is the exact same thing as giving a function $f_\alpha : W \to X_\alpha$ for every $\alpha \in \mathcal{A}$.

Theorem 15.3.5 (Universal property of direct products (of possibly infinitely many spaces))**.** Let $\mathcal{A}$ be a set and for each $\alpha \in \mathcal{A}$, choose a topological space X_α. Then there exists a topology on the direct product $\prod_{\alpha\in\mathcal{A}} X_\alpha$ satisfying the following properties:

(i) For every $\alpha \in \mathcal{A}$, the projection map $p_\alpha : \prod_{\alpha\in\mathcal{A}} X_\alpha \to X_\alpha$ is continuous.

(ii) For any topological space W, fix for every α a continuous function $g_\alpha : W \to X_\alpha$. Then there exists a continuous function $g : W \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ such that
$$g_\alpha = p_\alpha \circ g.$$

(iii) Moreover, g is unique among such functions. That is, if there is some other continuous function $h : W \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ such that, for all α, $g_\alpha = p_\alpha \circ h$, then $h = g$.

Again, this theorem is meant to make $\prod_{\alpha\in\mathcal{A}} X_\alpha$ seem *easier*. Even if you don't understand the space $\prod_{\alpha\in\mathcal{A}} X_\alpha$, you can understand all continuous functions into it. To give some continuous function to $\prod_{\alpha\in\mathcal{A}} X_\alpha$, you just need to give $\mathcal{A}$-many continuous functions $g_\alpha : W \to X_\alpha$. In the exercises, you will prove that every function to $\prod_{\alpha\in\mathcal{A}} X_\alpha$ arises in this way.

Definition 15.3.6. Let $\mathcal{A}$ be a set and for each $\alpha \in \mathcal{A}$, choose a topological space X_α. Then the *product topology* on the set $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is defined as follows: A subset $U \subset \prod_{\alpha\in\mathcal{A}} X_\alpha$ is called open if and only if U can be written as a union of sets of the form
$$\prod_{\alpha\in\mathcal{A}} V_\alpha$$
$V_\alpha \subset X_\alpha$ is open for each α, and $V_\alpha = X_\alpha$ for all but finitely many α.

Theorem 15.3.7. The product topology satisfies the properties of the theorem above.

Proof. (i) For every α, p_α is continuous, as for any open subset $U_\alpha \subset X_\alpha$, we see that $p_\alpha^{-1}(U_\alpha)$ is a direct product of all the $X_{\alpha'}$ (for $\alpha' \neq \alpha$) with U_α. Such a direct product is open in the product topology by definition.

(ii) Given continuous functions $g_\alpha : W \to X_\alpha$ for every α, define $g(w) = (g_\alpha(w))_{\alpha\in\mathcal{A}}$. Then it follows that for all α, $p_\alpha \circ g = g_\alpha$. We must now prove that g is continuous. To see this, let $U \subset X \times Y$ be open. Then
$$U = \bigcup_{\beta\in\mathcal{B}} U_\beta$$

where each U_β is of the form $\prod_{\alpha\in\mathcal{A}} V_\alpha$ and all but finitely many α satisfy the property that $V_\alpha = X_\alpha$, while all $V_\alpha \subset X_\alpha$ are nevertheless open. (This is by definition of product topology.)

Thus

$$g^{-1}(U) = g^{-1}\left(\bigcup_{\beta\in\mathcal{B}} U_\beta\right). = \bigcup_{\beta\in\mathcal{B}} g^{-1}(U_\beta).$$

Now we note that, for all $\beta \in \mathcal{B}$,

$$g^{-1}(U_\beta) = g_{\alpha_1}^{-1}(V_{\alpha_1}) \cap \dots g_{\alpha_n}^{-1}(U_{\alpha_n})$$

where the $V_{\alpha_1}, \dots, V_{\alpha_n}$ are the finitely many open subsets that are proper open subsets (of the $X_{\alpha_1}, \dots, X_{\alpha_n}$). This is a finite intersection of open subsets of W (because each g_α is continuous) hence for all β, the set $g^{-1}(U_\beta)$ is open in W. We thus see $g^{-1}(U)$ is a union of of open subsets of W, hence is open.

This proves g is continuous.

(iii) If h is another such function, then for all w and for all α, we see that $p_\alpha(h(w)) = p_\alpha(g(w))$. It follows that $h(w) = g(w)$ for all w, hence $h = g$. □

15.4 Exercises

Exercise 15.4.1. Show that a function $f : W \to X \times Y$ is continuous if and only if $p_X \circ f$ and $p_Y \circ f$ are both continuous.

Exercise 15.4.2. Show that the product topology is the *smallest* topology for which p_X and p_Y are both continuous. That is, if any other topology $\mathcal{T}'$ on $X \times Y$ satisfies the property that p_X and p_Y are continuous, then the product topology is contained in $\mathcal{T}'$.

Exercise 15.4.3. Let P and Q be posets and give each the Alexandroff topology. Let $\mathcal{T}$ be the product topology on $P \times Q$. On the other hand, recall that we can endow $P \times Q$ with a partial order relation by declaring that $(p, q) \le (p', q') \iff p \le p' \& q \le q'$. Let $\mathcal{T}'$ be the Alexandroff topology on $P \times Q$. Show that $\mathcal{T} = \mathcal{T}'$.

Exercise 15.4.4. Let $\mathcal{A}$ be a set and for each $\alpha \in \mathcal{A}$, suppose we are given a topological space X_α. For any topological space W, show that a function $f : W \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ is continuous if and only if, for each $\alpha \in \mathcal{A}$, $p_\alpha \circ f$ is continuous.

Exercise 15.4.5. Let $X_1, \ldots, X_n$ be a finite collection of discrete topological spaces. Show that the product space

$$X_1 \times \ldots \times X_n$$

is discrete.

Exercise 15.4.6. Let $\mathcal{A}$ be an infinite set. For each $\alpha \in \mathcal{A}$, let $X_\alpha = \{a, b\}$ with the discrete topology. Show that the product space

$$\prod_{\alpha \in \mathcal{A}} X_\alpha$$

is *not* discrete.

Exercise 15.4.7. Let $\mathcal{A}$ be an infinite set. For each $\alpha \in \mathcal{A}$, suppose we are given a space X_α with the trivial topology. Show that the product space

$$\prod_{\alpha \in \mathcal{A}} X_\alpha$$

also has the trivial topology.

Exercise 15.4.8. Let $\mathcal{A} = \mathbb{Z}_{\geq 1}$ be the set of positive integers, and for each $i \in \mathcal{A}$, let $X_i = \{0, 1, \ldots, 9\}$ be the set of digits (i.e., whole numbers between 0 and 9). One can think of an element of X_i as a "decimal" number whose unit digit is 0, whose 1st place past the decimal point is given by $x_1 \in X_1$, whose 2nd place past the decimal point is given by $x_2 \in X_2$, and so forth:

$$(x_1, x_2, x_3, \ldots) \in \prod_{i \in \mathbb{Z}_{\geq 1}} X_i \iff 0.x_1x_2x_3 \ldots.$$

This defines a function

$$f : \prod_{i \in \mathbb{Z}_{\geq 1}} X_i \to \mathbb{R}$$

by treating a decimal string as a real number.

(i) Is this function continuous?

(ii) Can you write down an equivalence relation on $\prod_{i \in \mathbb{Z}_{\geq 1}} X_i$ that makes f a bijection onto the subset $[0, 1] \subset \mathbb{R}$?

(iii) Is the inverse function $[0, 1] \to (\prod_{i \in \mathbb{Z}_{\geq 1}} X_i)/\sim$ continuous?