Lecture 17

Hausdorffness

So far we have seen many examples of topological spaces.

1. Any set with the discrete topology (every set is open)
2. Any set with the trivial topology (only the set itself and the empty set are open)
3. Posets with the Alexandroff topology (upward closed sets are open)
4. Euclidean space with the standard topology (a set is open if it's a union of open balls)
5. Subsets of spaces (given the subspace topology)
6. Quotients of spaces
7. Products of spaces.

This is a really long list! We haven't even scratched the surface on how many new spaces we can create using these tools.

But now that we're beginning to see many kinds of spaces, it will be good to know about *properties* that distinguish different spaces.

Example 17.0.1. For example, the poset $(\mathbb{R}, \geq)$ with the Alexandroff topology—is it homeomorphic to $\mathbb{R}$ with the standard topology? Obviously $\mathbb{R}$ is in bijection with itself (in many ways; but certainly via the identity function) but are any of the bijections homeomorphisms?

Does $(\mathbb{R}, \geq)$ have a topological property that $\mathbb{R}$ with the standard topology doesn't have? Or vice versa? If so, the two spaces could not be homeomorphic.

One example of a topological property we've seen is *compactness*. This is still something we need to develop intuition about, but if X is compact and Y isn't, then X and Y are not homeomorphic. So they're not equivalence spaces.

Today, we'll learn a new topological property, called Hausdorffness.

17.1 Interlude on learning math

I was inspired by one of your peers' homework assignments. They wrote the following:

Writing Assignment 6
October 2, 2020

After looking over my lecture notes for this class, the first thing that sticks out to me is the definition of what a topology is. After reading the definition, it seems to make sense and is easy for me to follow the requirements for a topology on X. However, a few remarks later, I come across an example containing posets and topologies. This example is where I find myself confused and find it difficult to understand what a topology really is. So, I go back into my notes and try to find the definition of what a poset is. After reviewing, I move back to the example and still find myself stuck and unsure how to go about this problem and realize that I don't really understand the definition of topology. What I think is confusing me the most is the actual word: topology. It is something I have never heard of and had no idea it had anything to do with mathematics until taking this course.

After rereading my notes, I decided to look up the definition of topology online. What I came across on several different websites are very similar to the definition given in class. However, one website in particular made things a little more clear. This PDF defines what a topological space is right before defining a topology. Being able to visualize what a topological space is before trying to figure out what a topology is actually cleared up some confusion I had. I understand now that a topological space (X, T), where X is a set and T is a set of subsets of X (satisfying certain axioms) and is a topology. Then I can clearly see that a topology T on a set X consisting of subsets of X satisfies 3 properties, which I am very clear on. Looking back, I think what really freaked me out is the actual word topology. It does feel more comfortable but I think I need to see more examples on how this works to feel 100% confident with it now. Before this assignment I was very unsure of what a topology is but now I feel that I could define what a topology is without being hesitant.

I am sure many of you can resonate with your peer's experience. It is universal! This is what learning math feels like; it's what good science feels like. Science is the exploration of things we don't understand, and understanding requires clear and precise thinking processes. That's what any good math class is trying to teach you.

You might know that the Fields Medal is one of the most prestigious prizes one can win in math. It's sometimes compared to the Nobel Prize. (There are no Nobel Prizes in math.) Here's what a Fields Medalist named Kunihiko Kodaira once said about reading and learning math. (Hiro translated this from Japanese.)

To me, there is nothing harder to read than a math book (papers included). To read through a math book of hundreds of pages, from beginning to end, is a Herculean task. When I open a math book, there are first axioms and definitions; then there are theorems and proofs. I know that mathematics is a thing which becomes incredibly easy and clear once you understand it, so I try to read only the theorems and somehow understand. I try to think of proofs on my own. Most of the time, I don't get it even after I think about it. Having no other choice, I try reading the proof in the book. But even after reading it once or twice, I still don't feel like I understand it. So I try copying the proof into my notebook. Then I notice a part of the proof I dislike. I try to think if there must be another proof. It's great if I find one right away, but otherwise it takes a long time until I give up. And if I go about in this way, after a month finally arriving at the end of the first chapter, I forget the content toward the beginning. Having nothing else I can do, I review the chapter from the start again. Then the entire structure of the chapter begins to bother me. I think things like, it seems better to take care of Theorem Seven before proving Theorem Three. So I create another notebook where I reorganize the whole chapter. I finally feel like I understand the first chapter, and I feel at peace, but it's troublesome that it took so terribly long. To get to the last chapter of a book with hundreds of pages is near impossible. I would very much appreciate it if somebody could teach me a quick way to read mathematical texts.

Of course, there is no quick way. Understanding is the one thing that nobody else can do for you, and importantly, that you have to do for yourself through great investment of time and patience.

17.2 Hausdorffness

Back to our regularly scheduled programming.

In Euclidean space, we've talked about how an open subset U contains "wiggle room" for all its elements. There's no obvious notion of "wiggle room" in other topological spaces (which might not have notions of distance), but regardless, let's pretend that a set being open is a proxy for the property that the set's elements have some wiggle room.

Question: Given two points in a topological space, can they each have some wiggle room to themselves?

This is an informal question. If any pair of points *can* be given wiggle room to their own, the space will be called Hausdorff (formal definition below). As we will see, not every space is Hausdorff, so this is a meaningful property of topological spaces.

Definition 17.2.1. Let X be a topological space. X is called *Hausdorff* if it satisfies the following property: For any pair of points $x, x' \in X$ with $x \neq x'$, there exists open subsets U, U' of X for which

- $x \in U$
- $x' \in U'$, and
- $U \cap U' = \emptyset$.

17.3 Examples of Hausdorff and non-Hausdorff spaces

Example 17.3.1. The main example you should keep in mind of a Hausdorff space is $\mathbb{R}^n$ (with the standard topology). Let's verify that $\mathbb{R}^n$ is Hausdorff.

Given $x, x' \in \mathbb{R}^n$ with $x \neq x'$, we know that the distance $d(x, x')$ is non-zero. So let's choose

$$U = \text{Ball}(x, d(x, x')/2)$$

and

$$U' = \text{Ball}(x', d(x, x')/2).$$

These are open balls of equal radius, but they do not intersect each other! (Can you see why? Hint: Triangle inequality.) In other words, the intersection of U and U' is the empty set. At the same time, clearly $x \in U$ and $x' \in U'$.

Because we can produce such U and U' for any x, x' with $x \neq x'$, we have shown that $\mathbb{R}^n$ (with the standard topology) is Hausdorff.

Remark 17.3.2. Hausdorffness is a kind of "separation axiom." "Axiom" is a word that was more en vogue in the past, and now-a-days we would probably call this a separation "property." An *axiom* is something we demand to be true, or take for granted to be true; but we won't be demanding that our spaces are Hausdorff.

Hausdorffness is a "separation" property because it asks whether we can "separate" points x and x' from each other using enough wiggle room. (The existence of open subsets, each containing x and x', that don't intersect, is a proxy for saying that both x and x' admit wiggle room that don't get in each other's ways.)

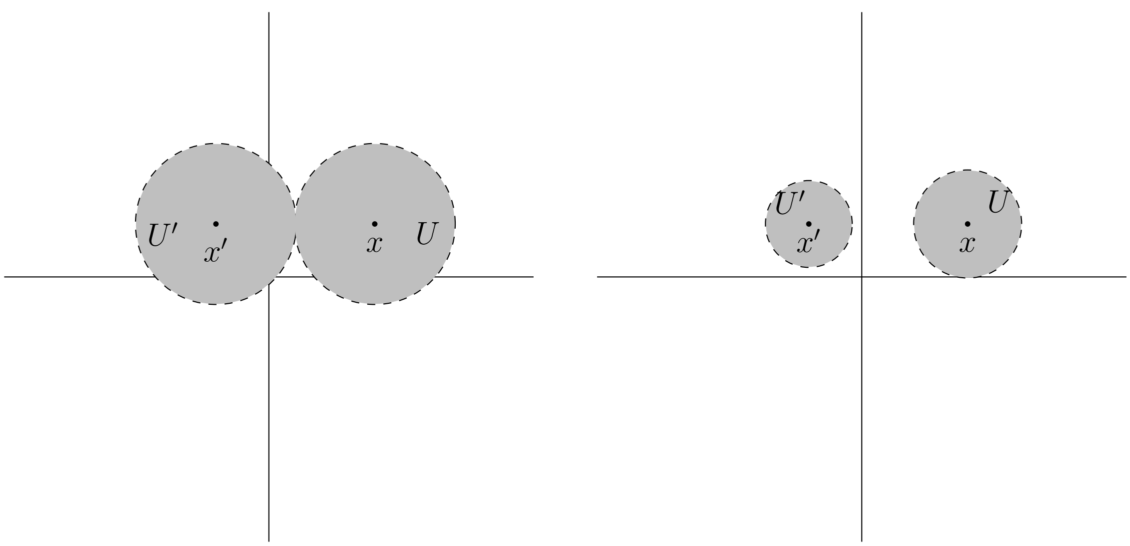

Figure 17.1: On the left, two open sets U and U' "separating" x from x'. On the right, another possible choice for U and U' (for the same x and x' as on the left).

Example 17.3.3. Let $P = [n] = \{0, 1, \ldots, n\}$ be our favorite poset. Then P is not Hausdorff. For example, if $x = 0$ and $x' = 1$, you cannot find any open subset of x that does not intersect an open subset of x'. (In fact, any open subset of x *contains* x'.

Example 17.3.4. let $P = \mathcal{P}(\{a, b\})$ be the power set of the two-element set $\{a, b\}$. Then P is not Hausdorff. For example, let $x = \{a\}$ and $x' = \{b\}$. then (by definition of the Alexandroff topology) any open set U containing x must also contain $\{a, b\}$. Likewise, any open set U' containing x' must contain $\{a, b\}$. Hence it is impossible to find U and U' (satisfying $x \in U$ and $x' \in U'$) for which $U \cap U'$ is empty.

So, if you thought that posets (with the Alexandroff topology) were strange topological spaces, the Hausdorff property justifies your opinion. In some ways, Hausdorff spaces tend to be "more intuitive" than non-Hausdorff spaces. On the other hand, a space like Euclidean space may be more familiar to us, and indeed, it is Hausdorff.

Example 17.3.5. Let X be a set. Then X with the discrete topology is Hausdorff.

If X is given the trivial topology, then X is Hausdorff if and only if X has one or fewer elements.

Example 17.3.6. Let X be Hausdorff. Then for any subset A of X, we can conclude that A is Hausdorff (when endowed with the subspace topology).

To see this, choose two distinct points $a, a' \in A$. (Distinct means $a \neq a'$.) Well, we know X is Hausdorff, so we can find $V, V' \subset X$ for which V, V' are both open, $a \in V, a' \in V'$, and $V \cap V' = \emptyset$.

Consider the sets $U = A \cap V$ and $U' = A \cap V'$. Each is open in A by definition of subspace topology. Clearly $a \in U$ and $a' \in U'$, and $U \cap U' \subset V \cap V' = \emptyset$, so $U \cap U' = \emptyset$. This shows that A is Hausdorff.

Example 17.3.7. If X and Y are homeomorphic, then X is Hausdorff if and only if Y is.

Example 17.3.8. Let X and Y be Hausdorff. Then the product space $X \times Y$ is Hausdorff. To see this, suppose that $(x, y) \neq (x', y')$. (This means that either $x \neq x'$, $y \neq y'$, or both.) Without loss of generality, suppose that $x \neq x'$. Then because X is Hausdorff, we can find open $U, U' \subset X$ so that $U \cap U' = \emptyset$ while $x \in U$ and $x' \in U'$.

Then choose any open subset $V \subset Y$ containing y, and any open subset $V' \subset Y$ containing y'. (We need not assume that $V \cap V' = \emptyset$.) Then $U \times V \cap U' \times V' = \emptyset$ because $U \cap U' = \emptyset$.

(Of course, by definition of product topology, both $U \times V$ and $U' \times V'$ are open.)

Warning 17.3.9. Let X be Hausdorff, and R an equivalence relation on X. Then it is possible that $X/\sim$ is *not* Hausdorff.

For example, let $X = \mathbb{R}$ with the standard topology, and define an equivalence relation $\sim$ for which

$$x \sim x' \iff \begin{cases} x \neq 0 \& x' \neq 0 & \text{or} \\ x = x' = 0. \end{cases}$$

Then $X/\sim$ is homeomorphic to the poset [1] (with the Alexandroff topology); I encourage you to exhibit this homeomorphism. On the other hand, [1] is not Hausdorff by a previous example.

Here is one way to summarize the previous example and warning: Hausdorffness is a property preserved by products and subspaces, but not by quotients.

17.4 Compact subspaces of Hausdorff spaces

Here is one way in which Hausdorff spaces behave somewhat similar to Euclidean spaces: Some slight sliver of the Heine-Borel theorem holds for Hausdorff spaces.

Proposition 17.4.1. Assume X is Hausdorff.

If $A \subset X$ is compact (when given the subspace topology), then A is a closed subset of X.

This is the best result we could hope for. There's no notion of "bounded" because X need not have any notion of distances. And the converse of the above has to be false—even if A is closed in X, A need not be compact. (For example, if $A = X = \mathbb{R}^n$.)

Proof. Let me begin with a claim:

Claim. *For any $y \in X \setminus A$, we can find an open subset U_y of X for which (i) $y \in U_y$ and (ii) U_y does not intersect A.*

Suppose that you believe this claim. Then we see that $A^C = \bigcup_{y \in A^C} U_y$. It is clear that $A^C \subset \bigcup_{y \in A^C} U_y$, while on the other hand, none of the U_y intersect A, so $\bigcup_{y \in A^C} U_Y \subset A^C$. In other words, A^C is open because it is a union of open sets. This proves that A is closed.

So everything is contingent on the above claim. This is where we use that X is Hausdorff and that A is compact.

Fix y. for every $x \in A$, choose open sets U_x and V_x for which $x \in U_x$ and $y \in V_x$, and $U_x \cap V_x = \emptyset$. (This is possible because X is Hausdorff.) Then the collection $\{U_x \cap A\}_{x \in A}$ forms an open cover of A. Because A is compact, we may choose a finite subcover

$$\{U_{x_1} \cap A, \ldots, U_{x_n} \cap A\}.$$

Note define $U_y = V_{x_1} \cap \ldots \cap V_{x_n}$. Note that U_y does not intersect any of the U_{x_i}. In particular, $U_y \cap A = \emptyset$ because the $A \subset \bigcup_{i=1,\ldots,n} U_{x_i}$. Importantly, U_y is open in X because it is a. *finite* intersection of open sets. This proves the claim. □

17.5 A convenient homeomorphism criterion

We've got some powerful adjectives at our disposal—compact, and Hausdorff. We'll use both in the following:

Theorem 17.5.1. Let X be compact and Y Hausdorff. Then any continuous bijection $f : X \to Y$ is a homeomorphism.

In other words, under the hypotheses of the theorem, you do *not* need to check whether f^{-1} is continuous to verify that f is a homeomorphism.

This is a very powerful theorem for producing a lot of homeomorphisms, especially when X arises as a quotient space.

Here is a sample application of the theorem:

Example 17.5.2. Let $X = [0, 2\pi]/\sim$ where $\sim$ is the equivalence relation for which $0 \sim 2\pi$, $2\pi \sim 0$, and $t \sim t$ for all $t \in [0, 2\pi]$.

Consider the function

$$g : [0, 2\pi] \to \mathbb{R}^2, \qquad t \mapsto (\cos(t), \sin(t)).$$

Of course, g has image given by the unit circle. By the universal property of subspace topology, we have an induced continuous map

$$[0, 2\pi] \to S^1.$$

On the other hand, $t \sim t' \implies g(t) = g(t')$ because $g(0) = g(2\pi)$; so this induces a continuous map by the universal property of quotient spaces:

$$f : [0, 2\pi]/\sim \to S^1.$$

f is an injection and a surjection[1]; moreover, the domain of f is compact because it is the quotient of a compact space. And the codomain is Hausdorff because it is a subspace of a Hausdorff space. So the theorem applies.

In short, f is a homeomorphism. This proves that $[0, 2\pi]/\sim$ is homeomorphic to S^1.

The proof of the theorem requires a lemma.

Lemma 17.5.3. Let X be compact. If $A \subset X$ is closed, then A (with the subspace topology) is compact.

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of A. We must exhibit a finite subcover.

By definition of subspace topology, for every α, there exists V_α open in X for which $U_\alpha = V_\alpha \cap A$.

Then the collection

$$\{A^C\} \cup \{V_\alpha\}_{\alpha \in \mathcal{A}}$$

is a cover of X. (We see that $A \subset \bigcup V_\alpha$ while $A^C \cup A = X$.) It is in fact an open cover because each V_α is open, and A^C is open by the assumption that A is closed.

Because X is compact, we may find a finite subcover, which may look like

$$\{A^C, V_{\alpha_1}, \ldots, V_{\alpha_n}\}.$$

In fact, the subcover may not contain A^C; in any case, let's consider the collection *without* A^C:

$$\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}.$$

[1] I encourage you to check this if you haven't seen this before!

We see that
$$A \subset A \cap \left(\bigcup_{i=1,\ldots,n} V_{\alpha_i} \right) = \bigcup_{i=1,\ldots,n} A \cap V_{\alpha_i}.$$
Hence the sets $U_{\alpha_i} = A \cap V_{\alpha_i}$ form a cover of A, and $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ exhibits the finite subcover we seek. □

Proof of Theorem 17.5.1. Let g be the inverse function of f. We must show that g is continuous. For this, it suffices to show that for every closed subset $A \subset X$, we have that $g^{-1}(A)$ is closed in Y.

Because g is the inverse to f, we see that $g^{-1}(A) = f(A)$. Because A is closed in X and X is compact, we see that A is compact by the lemma. We proved in a previous class (Proposition 10.5.1) that the continuous image of a compact space is compact, so $f(A)$ is compact. By Proposition 17.4.1, this shows that $f(A) = g^{-1}(A)$ is closed in Y. This shows that g is continuous, and we are done. □

17.6 Take-aways

It is easy to get lost in the weeds. A lot went into the above proofs. But here's what you need to walk away with:

- The definition of "Hausdorff."
- All the examples and non-examples of Hausdorff spaces.
- Theorem 17.5.1, which allows you to prove that certain continuous bijections are in fact homeomorphisms.