Lecture 18

Metric spaces

We've studied new constructions of spaces, and properties of spaces.

- Basic examples: Posets, Euclidean space, trivial topology, discrete topology.
- Ways to make new spaces: Subspaces, quotient spaces, products.
- Properties: Compactness, Hausdorffness.

Remark 18.0.1. If this were a course about "numbers," you could make a similar list as follows:

- Basic examples: 0, 1, π.
- Ways to make new numbers: Adding, multiplying, subtracting, dividing.
- Properties: Rational, irrational, positive, negative.

The ideas listed above are, of course, all central to your use and understanding of numbers. If you go on to deal with topological spaces in any form, the ideas we've learned so far are also central.

Today, we're going to learn about a new family of spaces. Metric spaces. These are, informally, spaces that arise by first cooking up some notion of *distance*.

And, just like Hausdorff spaces are "nice" (as we discussed last time), metric spaces will be even nicer.

18.1 Metrics

A *metric* is a way to define a notion of distance. The concrete definition will be given shortly, but let's write down some bare minimum requirements for what we mean by a distance between two points.

1. If x and x' are two points in a space with some notion of distance, if the distance between x and x' is zero, certainly x should equal x'. Conversely, if x and x' are the same point, the distance between them should be zero.

2. The distance from x to x' should be the same thing as the distance from x' to x.

3. Finally—though this requires some thought—if we know the distance from x to x'', then the sum of the distances through some intermediary point should be larger than that distance. In other words, the distance should represent some "minimal" way to get between two points, so if there is a third point x' involved, the distance from x to x', summed with the distance form x' to x'', should not be minimal, so should be larger than or equal to the distance from x to x''. (This wordy description will be made more succinct in the definition below.)

Great. Now let's make these requirements into checkable, mathematical conditions.

So how do we describe a "distance function?" This should be something which, to every pair of points in a set X, gives us a number called the distance between those two points.

In other words, a distance function ought to be a function from $X \times X$ to $\mathbb{R}$.

Definition 18.1.1 (Metric)**.** Let X be a set. A *metric* on X is a function

$$d : X \times X \to \mathbb{R}, \qquad (x, x') \mapsto d(x, x')$$

satisfying the following conditions:

1. For any $x, x' \in X$, we have that $d(x, x') = 0$ if and only if $x = x'$. (Non-degeneracy.)

2. For any $x, x' \in X$, we have that $d(x, x') = d(x', x)$. (Symmetry.)

3. For any $x, x', x'' \in X$, we have that

$$d(x, x') + d(x', x'') \geq d(x, x'').$$

   (The triangle inequality.)

18.2 Examples of metrics

We've already seen an example of a metric:

Example 18.2.1. Let $X = \mathbb{R}^n$, and define a function

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad (x, x') \mapsto \sqrt{\sum_{i=1}^{n} (x_i - x'_i)^2}.$$

This is the usual notion of distance on Euclidean space, and we call it the *standard metric* on $\mathbb{R}^n$. We have already seen that this satisfies the triangle inequality in past classes. I leave it to you to verify the other two conditions.

Here is another example:

Example 18.2.2. Let X be any set. The *discrete metric* on X is the function

$$d : X \times X \to \mathbb{R}, \qquad d(x, x') = \begin{cases} 1 & x \neq x' \\ 0 & x = x'. \end{cases}$$

In other words, all distinct points are declared distance 1 from each other.

So we've seen two examples of metrics on $\mathbb{R}^n$: the standard metric, and the discrete metric. There are many others!

Example 18.2.3. Let $X = \mathbb{R}^n$. The l^∞ metric on $\mathbb{R}^n$ is the following:

$$d_{l^\infty}(x, x') = \max_{i=1,\ldots,n} |x_i - x'_i|.$$

(Sometimes, this is also called the "sup metric.")

Example 18.2.4. Let $X = \mathbb{R}^n$. The taxi cab metric on $\mathbb{R}^n$ is the following:

$$d_{taxi}(x, x') = \sum_{i=1}^{n} |x_i - x'_i|.$$

These are just functions for now; we'll see ways in which they are different (and similar) later on.

Let me give you some more interesting examples.

Example 18.2.5. Let X be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. (Here, $[0,1]$ is given the usual subspace topology inherited from $\mathbb{R}$.) For this example, we will let letters like f and g denote elements of X. We will let t denote an element of $[0,1]$.

Then we define the following function:

$$d : X \times X \to \mathbb{R}, \qquad (f,g) \mapsto \int_0^1 |f - g|\, dt.$$

Yes, that is an integral. (Welcome back to calculus!) This is a really interesting example, because X is intuitively a gigantic set. (There are a *lot* of continuous functions from $[0,1]$ to the real line.) In many ways, it is *uncountably* infinite-dimensional. And this may be the first time in your life that you've thought about ways to put a notion of distance on such a gigantic set.

So let's carefully verify that d is indeed a metric.

(Non-degeneracy.) Suppose that $f - g$ does not equal zero. Then there is some $t_0 \in [0,1]$ for which $f(t_0) - g(t_0) \neq 0$. Because f and g are continuous, this means that there is some open interval containing t_0 on which $f - g$ is not zero. In particular, for some $\epsilon > 0$, we know that $f(t) - g(t)$ is never zero on the interval $[t_0 - \epsilon, t_0 + \epsilon]$. Thus $|f(t) - g(t)|$ is always positive on this interval. Using the the extreme value theorem, let m be the minimal value of $|f(t) - g(t)|$ on $[t_0 - \epsilon, t_0 + \epsilon]$. (Note that, necessarily, $m > 0$.) Then

$$\int_{t_0-\epsilon}^{t_0+\epsilon} |f(t) - g(t)|\, dt \qquad > \qquad \int_{t_0-\epsilon}^{t_0+\epsilon} m\, dt = m \cdot 2\epsilon > 0.$$

Of course, because $|f - g|$ is a non-negative function, we have that

$$\int_0^1 |f - g|\, dt \geq \int_{t_0-\epsilon}^{t_0+\epsilon} |f - g|\, dt > 2m\epsilon.$$

On the other hand, if $f-g$ does equal zero, of course $\int_0^1 |f-g|\,dt = \int_0^1 0\,dt = 10$.

(Symmetry.) This is clear, as $|f-g| = |g-f|$.

(Triangle inequality.) Note that for every $t \in [0,1]$, we have that $|f(t) - g(t)| + |g(t) - h(t)| \geq |f(t) - h(t)|$. (This is the usual triangle inequality for real numbers.) This means

$$d(f,g) + d(g,h) = \int_0^1 |f-g|\,dt + \int_0^1 |g-h|\,dt \tag{18.2.0.1}$$

$$= \int_0^1 |f-g| + |g-h|\,dt \tag{18.2.0.2}$$

$$\geq \int_0^1 |f-h|\,dt \tag{18.2.0.3}$$

$$= d(f,h). \tag{18.2.0.4}$$

Remark 18.2.6. By the way, you should remember that two functions f and g are equal if and only if $f(t) = g(t)$ for every element t in the domain. So make sure you understand the difference between expressions like f (which is a function) and expressions like $f(t)$ (which is a number) above.

18.3 Metrics are never negative

In the definition of metric, I said that the metric is a function $d : X \times X \to \mathbb{R}$. This might leave open the possibiliy that d could take on negative values, but in fact, we have:

Proposition 18.3.1. If $d : X \times X \to \mathbb{R}$ is a metric, then for every $x, x' \in X$, we have that$d(x,x') \geq 0$.

Proof. By the triangle inequality, we know that

$$d(x,x') + d(x',x'') \geq d(x,x'')$$

for any triple $x, x', x'' \in X$. So choose x'' to be equal to x. Then we have that

$$d(x,x') + d(x',x) \geq d(x,x).$$

By symmetry, the lefthand side becomes $2d(x,x')$. By non-degeneracy, the lefthand side is zero. hence

$$2d(x,x') \geq 0.$$

This shows that, for all $x, x' \in X$, $d(x,x')$ must be non-negative. □

18.4 Metric spaces

Definition 18.4.1. A *metric space* is a set X equipped with a metric d. We will sometimes write a metric space as a pair (X, d); we may also say "let X be a metric space" with the metric implicit in the notation.

18.5 Open balls

Definition 18.5.1. Let (X, d) be a metric space. Fix $x \in X$ and $r > 0$. Then the *open ball of radius* r *centered at* x is the set

$$\{x' \in X \mid d(x, x') < r\}.$$

We will denote the open ball centered at x of radius r by

$$\mathrm{Ball}(x, r).$$

Sound familiar? Yes, it's the exact same definition as for Euclidean space (with the standard metric), but what we see is that the notion of open ball makes sense for *any* metric space.

Example 18.5.2. Let X be any set and let d be the discrete metric. Then

$$\mathrm{Ball}(x, r) = \begin{cases} \{x\} & r \geq 1 \\ X & r > 1. \end{cases}$$

In other words, for the discrete metric, an open ball is either all of X, or a singleton set.

18.6 Topologies from metrics

Every metric space gives rise to a topological space.

Definition 18.6.1. Let (X, d) be a metric space. Then the *metric topology* on X, or the topology induced by the metric, is defined as follows. We say that $U \subset X$ is open if and only if U can be written as a union of open balls.

Sound familiar?

Example 18.6.2. Let $X = \mathbb{R}^n$ with the standard metric. Then the metric topology on $\mathbb{R}^n$ is equal to the standard topology on $\mathbb{R}^n$.

Example 18.6.3. Let X be any set and equip it with the discrete metric. Then the metric topology on X is equal to the discrete topology on X.

18.7 Sequences

Let (X, d) be a metric space. Then we can talk about convergence of sequences in X.

Definition 18.7.1. Fix a sequence $x_1, x_2, \ldots$ in X. We say that the sequence *converges* to a point $b \in X$ if the following holds: For every $\epsilon > 0$, there exists some integer N so that $i \geq N \implies d(x_i, b) < \epsilon$.

Again, sound familiar?

A lesson is that a bunch of things we've defined for Euclidean space only depended on some notion of distance (that is, on a metric). So once we have a metric, we can talk about open balls, sequences, et cetera, just as we might like to talk about in Euclidean space.

18.8 Continuity

We have a theorem as follows:

Theorem 18.8.1. Let (X, d_X) and (Y, d_Y) be metric spaces. (Here, d_X denotes a metric for X, and d_Y a metric for Y.) Fix a function $f : X \to Y$. Then the following are equivalent.

1. f is continuous (with respect to the metric topologies on X and Y).

2. For every $x \in X$ and for every $\epsilon > 0$, there exists $\delta > 0$ so that for all $x' \in X$,
$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon.$$

3. For every sequence $x_1, x_2, \ldots,$ in X converging to b, the sequence $f(x_1), f(x_2), \ldots$ converges to $f(b)$.

Just as with universal properties, this theorem is supposed to *help* you. Let me tell you why. This theorem gives you many, many different ways of seeing whether a function is continuous! That's a good thing. And you may ask, then, which of the three methods given by the theorem is the best strategy for proving that a function is continuous? As usual, this depends on how the spaces X and Y are defined, and also on how the function f is defined.

18.9 Exercises

Exercise 18.9.1. Let (X, d) be a metric space. Show that X (with the metric topology) is Hausdorff.

Exercise 18.9.2. Verify that the l^∞ metric is a metric.

Exercise 18.9.3. Draw the open ball of radius 1, centered at the origin, of d_{l^∞} on $\mathbb{R}^2$.

Exercise 18.9.4. Verify that the taxi cab metric is a metric.

Exercise 18.9.5. Draw the open ball of radius 1, centered at the origin, of d_{taxi} on $\mathbb{R}^2$.

Exercise 18.9.6. Let X be the set of continuous functions from $[0, 1]$ to $\mathbb{R}$, with the metric from Example 18.2.5.

Choose some $t_0 \in [0, 1]$. Consider the evaluation function

$$ev_{t_0} : X \to \mathbb{R}, \qquad f \mapsto f(t_0).$$

Is ev_{t_0} continuous?

Exercise 18.9.7. Let A be any set, and let (X, d) be any metric space. Let us call a function $f : A \to X$ *bounded* if there exists some real number M such that for every $a, a' \in A$, we have that $d(f(a), f(a')) < M$. Let $F(A, X)$ denote the set of all bounded functions from A to X.

Define a function

$$d_{\sup} : F(A, X) \times F(A, X) \to \mathbb{R}, \qquad (f, g) \mapsto \sup_{a \in A} d(f(a), g(a)).$$

(sup is the least upper bound; if you haven't taken an analysis class that has introduced sup yet, this exercise may be difficult.)

Is $d_{\sup}$ a metric?

Exercise 18.9.8. Let (X, d_X) and (Y, d_Y) be metric spaces. Define a function

$$d : (X \times Y) \times (X, Y) \to \mathbb{R}, \qquad ((x, y), (x', y')) \mapsto d_X(x, x') + d_Y(y, y').$$

Is d a metric? If so, how does the metric topology of $X \times Y$ compare to the product topology (when each of X and Y are given the metric topology)?

18.10 Take-aways

Why should we care about metric spaces?

First, metrics give us a new way to construct topological spaces. For example, it probably wasn't obvious at all that we can put a meaningful topology on the set of continuous functions on $[0, 1]$. But we can!

Second, the general results of this lecture shows a connection between certain kinds of *geometry* (when we can measure distances, we can do geometry) and *topology* (which has to do with very fuzzy notions like open sets). If you go forth with more math, this will not be the last time that you encounter such a connection.

Third, metrics give us *tools* for studying topological spaces and continuous functions. For example, if we know that X and Y have topologies that arise from metrics, we have three very different-looking ways to check whether a function $f : X \to Y$ is continuous.

Finally, metrics are more intuitive than abstract open sets. So you may find that metric spaces become your favorite kinds of spaces. For example, as you see in the exercises, any metric space is Hausdorff (hence nice).