Lecture 19

Isometries

One more definition about metric spaces. When are two metric spaces equivalent?

Definition 19.0.1. Let X and Y be metric spaces. A function $f : X \to Y$ is called an *isometry* if

(i) f is a bijection, and

(ii) For all $x, x' \in X$, we have that $d(f(x), f(x')) = d(x, x')$. [1]

Informally, f is a function that "preserves" the distances in the domain with the distances in the codomain.

Example 19.0.2. As an example, fix a real number a and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function sending a vector x to ax. This is an isometry if and only if $a = \pm 1$.

Your mathematical scope has expanded so much! We now know about sets (two of which are equivalent when there is a bijection between them), posets (two of which are equivalent when there is a poset isomorphism between them), spaces (two of which are equivalent when there is a homeomorphism between them), and metric spaces (two of which are equivalent when there is an isometry between them).

[1]Note that this equality involves the metric on Y and the metric on X.

19.1 Exercises

Exercise 19.1.1. (a) Let $f : X \to Y$ be an isometry. Show that the inverse function to f is also an isometry.

(b) Show that the composition of two isometries is an isometry.

Exercise 19.1.2. Is every isometry a homeomorphism?

Given two metric spaces X and Y, and a homeomorphism f from X to Y (giving each space the metric topology), must f be an isometry?

Exercise 19.1.3. Fix a real number θ and consider the matrix

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Show that the function $x \mapsto Ax$ is an isometry of $\mathbb{R}^2$ with the standard metric.

For what values of θ is this an isometry for the taxicab metric? For the l^∞ metric?