Lecture 20

Path-connectedness

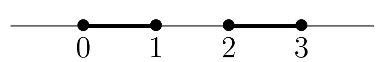
We are really building up some tools to study spaces. Today, we'll talk about the idea of what it means for a space to be *path-connected*. It's another nice property of a topological space. It will also lead to our first "invariant" of a topological space!

20.1 Path-connectedness

Let's say you're in a topological space X, at a point x_0. Is it possible to "continuously walk" from x_0 to x_1? Let's first try to define what such a continuous walk would be.

We begin with an example.

Example 20.1.1. Let $X = [0,1] \coprod [2,3] \subset \mathbb{R}$, drawn below:



Would you call X connected?

Remark 20.1.2 (Properties of spaces vs. properties of subsets)**.** Above, I used that X was a subset of $\mathbb{R}$ to define the topology of X, but once we know about X's topology, we could ask the connectedness question of X (without reference to $\mathbb{R}$). Is the following space connected?

(Importantly, the picture makes no reference to $\mathbb{R}$ itself.) So unlike "closed" or "open," the adjective "connected" makes sense as a property of a space X. And when we ask whether a subset is connected, we are asking about the property of that subset as a space (endowed with the subspace topology). Aside from specifying the topology of the subspace, the parent set is irrelevant to the question of connectedness.

I want to talk today about two different ways to talk about the *connectedness* of a topological space.

This is the most intuitive definition. First, some preliminaries: We let

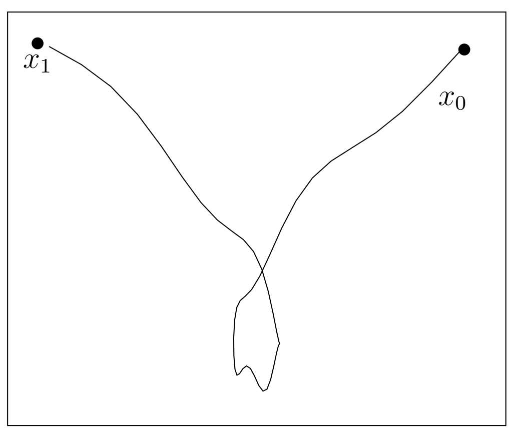
$$[0, 1]$$

denote the usual closed interval from 0 to 1. We treat it as a topological space by giving it the subspace topology inherited from $\mathbb{R}$.

Definition 20.1.3. Let X be a topological space. A *continuous path* (or *path* for short) in X is a continuous function

$$\gamma : [0, 1] \to X.$$

Example 20.1.4. Below is an image of a possible path $\gamma : [0, 1] \to \mathbb{R}^2$.



Note that a path need not be injective (it can cross over itself).

Definition 20.1.5. Let X be a topological space, and fix a path $\gamma : [0, 1] \to X$. We say that γ is a path *from* $\gamma(0)$ *to* $\gamma(1)$.

Definition 20.1.6. Let X be a topological space. We say that X is *path-connected* if for any two points $x, x' \in X$, there exists a path from x to x'.

We can straightforwardly check that path-connectedness is indeed a property of topological spaces preserved by homeomorphisms:

Proposition 20.1.7. If X and Y are homeomorphic, then X is path-connected if and only if Y is.

Proof. Suppose X is path connected. We must show Y is path-connected. So choose $y_0, y_1 \in Y$. We must exhibit a continuous path from y_0 to y_1. To do this, let $f : X \to Y$ be a homeomorphism, g the inverse to f, and let $x_i = g(x_i)$. Because X is path-connected, there is a continuous map $\gamma : [0,1] \to X$ satisfying $\gamma(0) = x_0$ and $\gamma_1 = x_1$. Because f is continuous, $f \circ \gamma$ is a continuous function from $f(x_0)$ to $f(x_1)$. We are finished by observing that $f(x_i) = y_i$.

If Y is path-connected, we can see that X is path-connected by the same argument. □

Remark 20.1.8. In fact, the proof method shows that if there is a continuous surjection from X to Y, then the path-connectedness of X implies the path-connectedness of Y.

20.2 Examples

Example 20.2.1. Let $X = \mathbb{R}$. Then X is path-connected. To see this, fix any two points $x, x' \in X$. Then define a function γ by "drawing a straight path from x to x'." The previous sentence was vague, so let's make it precise: Define

$$\gamma : [0,1] \to X, \qquad \gamma(t) = x + t(x' - x).$$

Note that x and x' are constants (we've fixed them!) while t is the variable.

γ is a continuous function. Let's shed some light on why: Because we've given $[0,1]$ the subspace topology, the inclusion

$$[0,1] \to \mathbb{R}, \qquad t \mapsto t$$

is a continuous function. Now let $f : \mathbb{R} \to \mathbb{R}$ be the linear function $t \mapsto x + t(x' - x)$. This is continuous (for example, it's a polynomial). Hence the composition

$$[0,1] \to \mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous. On the other hand, this composition is precisely γ.

Finally, note that $\gamma(0) = x$ and $\gamma(1) = x'$.

Example 20.2.2. More generally, let $X = \mathbb{R}^n$ (with the standard topology). Then X is path-connected. To see this, given x and x' in X, again define

$$\gamma : [0,1] \to \mathbb{R}^n, \qquad t \mapsto x + t(x' - x).$$

Note now that we are using vector scaling and vector addition/subtraction to define γ. This is continuous because because the standard topology on $\mathbb{R}^n$ is the product topology. By the universal property of the product topology, to check the continuity of γ, we only need to check that every component of γ is continuous. That is, we just need to check that the n functions

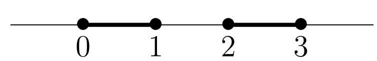
$$t \mapsto x_1 + t(x_1' - x_1), \qquad \ldots, \qquad t \mapsto x_n + t(x_n' - x_n)$$

are continuous. But we saw this already in the previous example!

So γ is continuous, and γ is a path from x to x'.

Let's also see some examples of spaces that are *not* path-connected. In the following examples, the main tool we use will be the *intermediate value theorem* from calculus.

Example 20.2.3. Let $X = [0,1] \coprod [2,3] \subset \mathbb{R}$, drawn below as before:



Then X is not path-connected.

Indeed, I'll take x to be some point in $[0,1]$ and x' to be some point in $[2,3]$. Suppose (for the purpose of contradiction) that there is a path

$$\gamma : [0,1] \to X$$

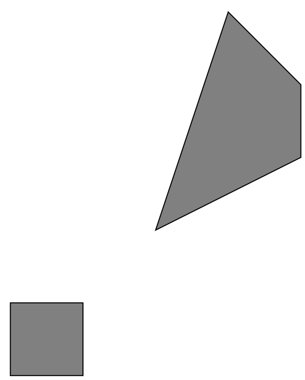
from x to x'. Then the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}$$

(where the second map is the inclusion map) is continuous. By the intermediate value theorem from calculus, for any value y such that $x \leq y \leq x'$, there must be some $t \in [0,1]$ such that $f(t) = y$.

But γ has image contained in X, and in particular, the composition f has no image in the open interval $(1,2)$. In particular, we have been led to a contradiction.

Example 20.2.4. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

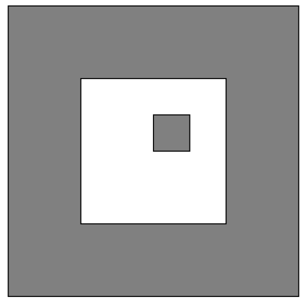


Then X is not path-connected. The proof is similar as the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from x to x', where x is in the lower-right component of X and x' is in the upper-left component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is the projection map sending $(x_1, x_2) \mapsto x_1$. Then f is continuous, being a composition of continuous functions; but again, f will violate the intermediate value theorem.

Example 20.2.5. Let X be the subset of $\mathbb{R}^2$ shaded below, given the subspace topology:



Then X is not path-connected. The proof is similar as the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from x to x', where x is in the middle component of X and x' is in the outer component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is now the map sending an element $y \in \mathbb{R}^2$ to the number $d(x,y)$. Then f again violates the intermediate value theorem.

(Here, we are using the very nice fact that $d(x,-)$ is a continuous function on any metric space.)

Remark 20.2.6. In studying path-connectedness, we may draw pictures or use arguments reminiscent of analysis class. This is because of the central role of the real line in these discussions ($[0,1]$ is a subspace of $\mathbb{R}$), and because your analysis class is devoted to the study of the real line.