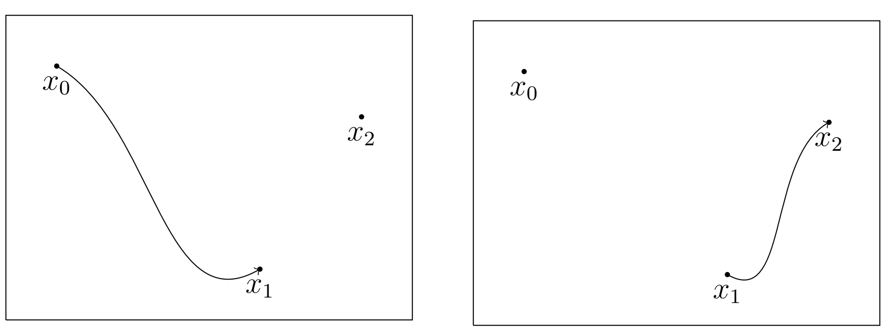
Lecture 21

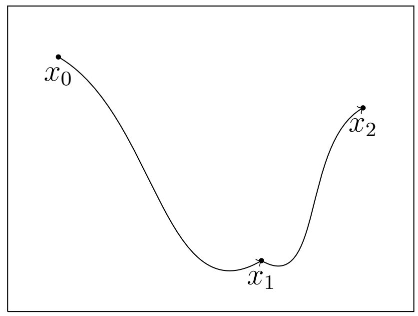
Invariance of domain and π_0

21.1 Concatenation of paths

Let X be a topological space, and choose three points x_0, x_1, x_2. Suppose you have a path from x_0 to x_1, and a path from x_1 to x_2.



Intuitively, it seems clear that there should be a path from x_0 to x_2. Indeed, why don't we just "do the first path, and then do the second?"



This is a great idea, but we need to turn the idea into a successful construction. Let's call this idea a "concatenation" of paths. What we need to do, then, is given two continuous paths that suitably agree at the endpoints, to construct a third called their concatenation. Here's how we'll define it.

Definition 21.1.1. Let $\gamma : [0,1] \to X$ and $\gamma' : [0,1] \to X$ be two continuous paths in X. Suppose that $\gamma(1) = \gamma'(0)$. Then the *concatenation* of γ with γ' is denoted

$$\gamma' \# \gamma$$

and is defined to be the function

$$t \mapsto \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \gamma'(2t-1) & t \in [1/2, 1]. \end{cases}$$

This is a bit much to parse, so let's talk it out. First, what does this function $\gamma' \# \gamma$ do when $t \leq 1/2$? It does exactly the path γ, but in double-time. (That is, twice as fast.) You might imagine a movie played on fast-forward, so that only the first half of the one-second allotted is used up to play the whole movie γ.

So what does the function do for $\geq 1/2$? The important thing to note here is that the function $t \mapsto 2t - 1$ is a bijection between $[1/2, 1]$ and $[0, 1]$. It sends $t = 1/2$ to 0, and sends $t = 1$ to 1. In other words, though $[1/2, 1]$ is an interval of only length 1/2, the function $t \mapsto \gamma'(2t - 1)$ "plays the entire movie of γ'" during the half-length time interval $[1/2, 1]$.

Informally, the concatenation $\gamma' \# \gamma$ "does γ at double-speed, then does γ' at double-speed." In particular, note that $\gamma' \# \gamma(0) = x_0$, and $\gamma' \# \gamma(1) = x_2$. And though we will not need this, note also that $\gamma' \# \gamma(1/2) = x_1$.

So now that we understand the concatenation as a function, we need to show that it is actually a path. That is, do we know that $\gamma' \# \gamma$ is continuous?

Proposition 21.1.2. Let $\gamma : [0,1] \to X$ and $\gamma' : [0,1] \to X$ be two continuous paths in X. Suppose that $\gamma(1) = \gamma'(0)$. Then $\gamma' \# \gamma$ is continuous.

Here is an informal reason as to why the proposition isn't obviously false: Because we know that $\gamma(1) = \gamma'(0)$, the concatenation $\gamma' \# \gamma$ doesn't have any "breaks" or "jumps." But this is informal. We'll need to actually prove that the concatenation is continuous using topology. (Note that X is an arbitrary topological space—it may not even be $\mathbb{R}^n$, and it could be some crazy metric space, or a crazy poset.)

To not lead ourselves astray, and to present the proof "the right way," we will delegate the proof to an add-on section at the end of this lecture note. You can just take Proposition 21.1.2 for granted.

Example 21.1.3. Let $m \geq 2$, and choose some point $x \in \mathbb{R}^m$. Let's prove that $X = \mathbb{R}^m \setminus \{x\}$ (that is, $\mathbb{R}^m$ with a point removed) is path-connected. (We are giving this set the subspace topology inherited from $\mathbb{R}^m$.)

Well, how did we prove $\mathbb{R}^m$ is path-connected? Given x' and x'', we defined a path by $(1-t)x' + tx''$. This is a fine, continuous path from x' to x'' in X so long as this line segment from x' to x'' never intersects x (i.e., so long as it doesn't pass through the point we removed). In other words, if x, x', x'' are not collinear, or if x does not lie along the line interval from x' to x'', we have proven that there is a path in X from x' to x''.

Now suppose that x, x', x'' lie along a single line in $\mathbb{R}^n$ and that x is between x' and x'' along this line. Because $m \geq 2$, we can find some point y that does not lie on the line between x' and x''.[1] But by the previous paragraph, there is a path from x' to y, and a path from x'' to y (because y, x', x'' is not collinear). By Proposition **??**, the concatenation of these two paths produces a path from x' to x''. This finishes the proof.

21.2 Application: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$

Have you asked the question where $\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$? Intuitively, we'd like to say yes; but we certainly haven't proven this in class. To give you

[1] This is the only place we are using the assumption on m.

some idea of the subtlety of this, note that there *do* exist bijections between $\mathbb{R}$ and $\mathbb{R}^2$.[2]

In fact, there are many bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ (even when $m \neq n$). In the earliest days of topology, we had the following basic question:

Question. (Invariance of domain.) If $m \neq n$, can $\mathbb{R}^m$ be homeomorphic to $\mathbb{R}^n$?

This is one of those basic questions that we think we ought to be able to answer. We can, now, but there were days when we couldn't! It's both an exciting and anxious time—we have a tool called topology, but can it even answer the most basic questions? It's a very good test of whether the notion of "open sets" and "homeomorphism" allow for intuitions of dimension, or whether we should scrap the whole idea of topological spaces and start anew. The proof that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$ often goes through some tools called the Brouwer fixed point theorem, or homology, but we won't cover those topics just yet. (We will not cover homology in this course.)

We prove a baby version of invariance of domain. It's an application of path-connectedness.

Theorem 21.2.1. $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}$ if and only if $m = 1$.

Proof. We'll prove that $m = 1$ is the only value of m for which $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}$.

Note that if $m = 0$, $\mathbb{R}^0$ is a point, so it's not even in bijection with $\mathbb{R}$.

Now suppose $m \geq 2$, and suppose $f : \mathbb{R}^m \to \mathbb{R}$ is a homeomorphism for the sake of contradiction. Choose a point $x \in \mathbb{R}^m$, and consider the space $U = \mathbb{R}^m \setminus \{x\}$ (given the subspace topology). Because f is a homeomorphism, we see that U is homeomorphic to $\mathbb{R} \setminus f(x)$.[3]

Note that $\mathbb{R} \setminus f(x)$ is not path connected. (This is a consequence of the intermediate value theorem.) Then Proposition 20.1.7 implies that U is

[2]This is a subtle fact that they might not always teach you in Math 3330. Here is one argument—$\mathbb{R}$ is in bijection with the set of subsets of $\mathbb{Z}_{>0}$. This shows that $\mathbb{R}^2$ is in bijection with the set of subsets of $\mathbb{Z}_{>0} \coprod \mathbb{Z}_{>0}$. But $\mathbb{Z}_{>0}$ is in bijection with $\mathbb{Z}_{>0} \coprod \mathbb{Z}_{>0}$, so $\mathcal{P}(\mathbb{Z}_{>0}) \cong \mathcal{P}(\mathbb{Z}_{>0} \coprod \mathbb{Z}_{>0})$.

[3]Here is the argument. The composition $U \to \mathbb{R}^m \to \mathbb{R}$ is continuous because composition of continuous functions is continuous. The composition has image $f(U)$, so by the universal property of the subspace topology for $f(U) \subset \mathbb{R}$, we obtain a continuous map $U \to f(U)$. Running the same argument for the inverse to f, to obtain a continuous map $f(U) \to U$, proves that $f(U)$ and U are homeomorphic.

not path-connected (because two homeomorphic spaces are either both path connected, or neither is path-connected). We have run into a contradiction, because Proposition 21.1.3 tells us that $\mathbb{R}^m \setminus \{x\}$ is path-connected. □

You can see that this proof method fails when $n \geq 2$, because then $\mathbb{R}^n \setminus \{x\}$ is still path-connected. One of the big victories of algebraic topology was the discovery of notions of "connectedness" the go beyond considering only paths, but also disks of higher dimensions. You might learn about these "higher homotopy groups" if you take my topics course next semester, or if you take an algebraic topology class.

Brouwer proved the invariance of domain theorem (that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$) in 1912.

21.3 Closed intervals and time reversal

By definition, the notion of path-connectedness depends on the topology of $[0, 1]$ (because we need to know which functions out of $[0, 1]$ are continuous). So let's study how we can think about $[0, 1]$, and other intervals, as a topological space, along with some cool things we can do with intervals.

We'll think about the interval $[0, 1]$ as parametrizing time. For example, we'll think of a continuous function

$$\gamma : [0, 1] \to X, \qquad t \mapsto \gamma(t)$$

as giving a point $\gamma(t)$ inside of X for every "time" $t \in [0, 1]$.

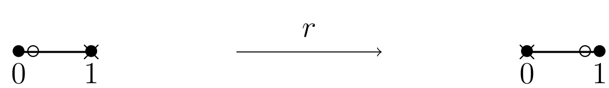
Proposition 21.3.1 (Time reversal is continuous)**.** Consider the function

$$r : [0, 1] \to [0, 1], \qquad r(t) = 1 - t.$$

1. r is continuous.

2. In fact, r is a homeomorphism.

Remark 21.3.2. The above proposition tells us that "reversing time" is a continuous operation, and that it can be undone. Here is a picture of r:

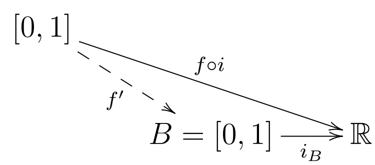


The markings (the open dot, the closed dot, and the closed square) indicate which points of the domain are sent to which points in the codomain.

Proof. (1) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = 1 - t$ is continuous. (An example proof: f is a polynomial, and polynomial functions are continuous. Another proof can be obtained by using ϵ-δ; any $\delta > 0$ with $\delta \leq \epsilon$ will do.)

By definition of subspace topology, the inclusion $i : [0,1] \to \mathbb{R}$ is continuous, so the composition $f \circ i$ is continuous.

You can check that $f \circ i$ has image given by $B = [0,1]$. Hence, by the universal property for the subspace topology of B, we have a continuous function as in the dashed arrow:



(Of course, B is equal to $[0,1]$, but I used the notation B to make clear *how* I was using the universal property.) This dashed arrow satisfies the property that $i_B \circ f' = f \circ i$. In other words,

$$f'(t) = i_B(f'(t)) = f(i(t)) = f(t) = 1 - t.$$

That is, $f' = r$. This shows r is continuous.

(2) Now, we can write down the inverse function to r straightforwardly (using algebra). For example, if a function s is an inverse to r, then

$$t = r(s(t)) = 1 - s(t)$$

so

$$s(t) = 1 - t.$$

(What we see is that r is its own inverse!) In particular, $s(r(t))$ also equals t, so s is both a right and left inverse to r. This proves that r is a bijection.

Finally, we know that r^{-1} is given by r, so from (1), we conclude that the inverse function to r is continuous. This shows that r is a homeomorphism. □

Remark 21.3.3. The method of proof for (1) extends more generally: If $f : \mathbb{R} \to \mathbb{R}$ is any continuous function, and if $A \subset \mathbb{R}$ is any subset, then the composition $f \circ i_A$ is continuous, and if $B = f(A)$, then the induced function from A to B is continuous (giving both A and B the subspace topology).

Example 21.3.4. Here is an application of the proposition. Let $\gamma : [0,1] \to X$ be a continuous function to some topological space X. We can think of this as a "continuous movie" of the point $\gamma(0)$ traveling to the point $\gamma(1)$.

Then $\gamma \circ r$ is also a continuous function to X (because r is continuous, and compositions of continuous functions are continuous).

In other words, if we can depict a "continuous movie" of $\gamma(0)$ traveling to $\gamma(1)$, then we can depict a continuous movie of the reverse; $\gamma(1)$ traveling to $\gamma(1)$ (along the "same path," but backward).

Proposition 21.3.5. Let X be a topological space, and fix $x, x' \in X$. If there exists a path from x to x', then there exists a path from x' to x.

This should be an intuitive proposition: If there's a path from x to x', you can just "reverse" the path to get from x' to x. That's the intuition we'll follow in the proof.

Proof. Let
$$\gamma : [0,1] \to X$$
be a path from x to x' (so $\gamma(0) = x$, and $\gamma(1) = x'$). Let us define
$$\overline{\gamma} = \gamma \circ r.$$
Because r and γ are continuous, the composition $\overline{\gamma}$ is. Moreover,
$$\overline{\gamma}(0) = \gamma(f(0)) = \gamma(1) = x'$$
and likewise, $\overline{\gamma}(1) = x$. Thus $\overline{\gamma}$ is a path from x' to x. □

Remark 21.3.6. Let X be a topological space. Then for any $x \in X$, there exists a path from x to itself. To see this, note that the constant path
$$\gamma : [0,1] \to X, \qquad \gamma(t) = x \forall t \in [0,1]$$
is a path from x to itself.

21.4 Path-connected components

Let's collect our knowledge about paths between points so far.

(i) Let $x \in X$. Then there is a path from x to itself. (For example, the constant path.)

(ii) Let $x, x' \in X$. If there is a path from x to x', then there is a path from x' to x. (For example, by time reversal. See Proposition 21.3.5.)

(iii) Let $x, x', x'' \in X$. If there is a path from x to x', and if there is a path from x' to x'', then there is a path from x to x''. (For example, by concatenation. See Proposition 21.1.2.)

All this is to say that there is an equivalence relation on any topological space X given as follows: We say $x \sim x'$ if and only if there exists a path from x to x'. Though we may not see this too often in this class, there is a name for the set of equivalence classes for this relation:

$$\pi_0(X) = X/\sim .$$

The left-hand side is read "pie nought of X." It is also called the set of "path-connected components" of X.

Add-on: Concatenating intervals, coproducts, and paths

Consider the interval $[0,2]$. You may not have noticed this before, but you can think of $K = [0,2]$ as "glued" out of the intervals $I = [0,1]$ and $J = [1,2]$ simply by identifying the element $1 \in I$ with the element $1 \in J$. The process of taking two intervals I and J, and gluing the right-endpoint of I to the left-endpoint of J to produce a new interval K, is called *concatenation.*

Now, the following caveat is both powerful and confusing. Even if I and J are the same interval, we can still consider what happens when we *treat* I and J as separate intervals, and glue I to J along the appropriate endpoints.

Example 21.4.1. Let $I = [0,3]$ and $J = [0,3]$. We can concatenate I and J to obtain an interval equivalent to the interval $[0,6]$. More generally, if $I = J = [0,t]$, then the concatenation of I and J results in an interval equivalent to $[0,2t]$.

This is secretly very important for us, but let me relegate its explanation to an add-on section for those of you who are curious–Section 21.5. I think the idea of concatenation is intuitive enough that we can move forward without needing to speak of coproducts.

21.5 Coproducts

The formal process of taking two sets, and ignoring whatever overlap they may have, is called taking the *disjoint union*, or taking the *coproduct*, of the two sets. We denote the disjoint union of I and J by

$$I \coprod J.$$

That symbol is literally an upside-down capital Pi.[4] The symbol $\coprod$ is also sometimes called the *coproduct* symbol. (It is also an upside down product symbol.[5])

We won't go too in-depth about coproducts in this rendition of the course, though you can look at course notes from a previous rendition of this course if you want more details.

The point of coproducts is to do something we can't easily do in physical reality. Suppose A and B are two sets, and that they happen to have some overlap (so $A \cap B \neq \emptyset$).

For our discussion, you might imagine that A is some region of a map, while B is another region, and the two regions overlap.

If you were to physically "cut out" A from your map, you'd of course take along some portion of B into the cut-out. ($A \cap B$ would be a subset of your cut-out.) In other words, you can't physically separate A from B.

What the operation of coproduct does is it "clones" both A and B. Here's what a coproduct would be in our physical example. Imagine somebody taking a photocopy of region A, and then a photocopy of region B. Then these photocopies are a copy of A and a copy of B that no longer physically overlap! This *photocopied* collection is how you can think about the coproduct of A and B.

So $A \cup B$ is the *actual union* of the regions A and B on the map. But $A \coprod B$ is the *union of the photocopies*. As an example, if $A \cap B$ contained Kazakhstan, then $A \coprod B$ would have *two* photocopied copies of Kashakstan, while $A \cup B$ would have only one.

In particular, note that in general,

$$A \cup B \neq A \coprod B.$$

[4] A capital Pi looks like Π (a big, rigid π).

[5] Recall from a previous class that we denote products using capital Pi: Π, just as in algebra, we denote sums using capital Sigma: Σ.

For example, if $A \cup B$ is a set with 5 elements, and if $A \cap B$ has 2 elements, then $A \coprod B$ has 7 elements.

Here is a rigorous definition in case you are curious:

Definition 21.5.1. Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of sets. The *coproduct*, or *disjoint union*, of this collection is the set

$$\{(x, \alpha) \mid x \in X_\alpha\} \subset \left(\bigcup_{\alpha \in \mathcal{A}} X_\alpha \right) \times \mathcal{A}.$$

The disjoint union is defined by

$$\coprod_{\alpha \in \mathcal{A}} X_\alpha.$$

Remark 21.5.2. The big set $(\bigcup_{\alpha \in \mathcal{A}} X_\alpha) \times \mathcal{A}$ can be thought of as taking the (usual) union of all the X_α, then taking $\mathcal{A}$-many photocopies of that union. Of course, you don't need photocopies of *entire* maps, but only of the specified regions X_α. So for every αth photocopy, the disjoint union only contains those x in the αth photocopy that are contained in X_α.

And, we can define a topology on the coproduct, called the *coproduct topology*.

21.5.1 Concatenating intervals, revisited

Definition 21.5.3. Let $I = [a, b]$ and $J = [a', b']$ be two closed intervals. Then the *concatenation* of I with J is the topological space

$$\left(I \coprod J \right) / \sim,$$

where $\sim$ is the equivalence relation given as follows:

$$s \sim t \iff \begin{cases} s = t & \text{or} \\ s = b \in I \& t = a \in J & \text{or} \\ t = b \in I \& s = a \in J & . \end{cases}$$

(In words, we are gluing the rightmost endpoint of I to the leftmost endpoint of J.) Note that we are treating I and J as non-overlapping photocopies when we take the coproduct.

Proposition 21.5.4. The concatenation of I with J is homeomorphic to an interval of length $(b - a) + (b' - a')$.

21.5.2 All closed intervals of positive length are homeomorphic

Intuitively, two closed intervals look identical; but they may have different lengths. Below may be a first hint about how topologies do *not* care about geometric measurements like length, but only care about notions like shape. The Proposition states that any two closed intervals are always homeomorphic (so long as they each have positive, finite length).

Proposition 21.5.5. Let $[a, b]$ and $[a', b']$ be two closed intervals, and assume that $a - b \neq 0, b' - a' \neq 0$. [6]

Then $[a, b]$ and $[a', b']$ are homeomorphic.

Proof. We have a linear function f sending a to a' and b to b'. [7] The formula for f is

$$f(t) = \frac{b' - a'}{b - a}(t - a) + a'.$$

(Note that $b-a \neq 0$ by assumption.) You can check that the interval $[a, b]$ has image $[a', b']$ under this function, so the same techniques as in the previous proof (see Remark 21.3.3) shows that f is a continuous function from $[a, b]$ to $[a', b']$.

On the other hand, we can produce an inverse to f. Let's call it g. The formula is

$$g(t) = \frac{b - a}{b' - a'}(t - a') + a.$$

(One way to reason out this formula quickly: g should be the linear function taking a' to a, and taking b' to b.) The same arguments as before show that g defines a continuous function from $[a', b']$ to $[a, b]$. ☐

[6] Here, $a, a', b, b' \in \mathbb{R}$. In particular, all intervals are finite-length. Note also that every interval has some *positive*—i.e., non-zero—length.

[7] The two points (a, a') and (b, b') determine a line in $\mathbb{R}^2$; this line is the graph of f.