Lecture 22

Connectedness. Stereographic projection

Today we're going to talk about a different, more abstract notion of connectedness. Then we'll talk about a very useful function to know about: Stereographic projection. This will lead us into the topic of one-point compactifications.

22.1 Being open and closed in $[0,1]$

For reasons that aren't obvious, let's see something interesting about the topology of $[0,1]$:

Proposition 22.1.1. Suppose that $A \subset [0,1]$ is a subset which is both closed and open. Then A is either empty, or equal to $[0,1]$.

For this, we'll use a Lemma:

Lemma 22.1.2. If $B \subset [0,1]$ is open, and if $b \in B$ does not equal 0 or 1, then there exists some $\epsilon > 0$ so that $(b-\epsilon, b+\epsilon) \subset B$.

Proof of Lemma 22.1.2. Since $B \subset [0,1]$ is open, by definition of subspace topology, there exists $W \subset \mathbb{R}$ open so that $B = W \cap [0,1]$. Now consider the intersection $W \cap (0,1)$. This is an open subset of $\mathbb{R}$, being the intersection of two open subsets—in particular, for any $b \in W \cap (0,1)$, there exists an open ball fully contained in $W \cap (0,1)$ containing b. Let ϵ be the radius of this open ball. Then

$$(b-\epsilon, b+\epsilon) = \text{Ball}(b;\epsilon) \subset W \cap (0,1) \subset W \cap [0,1] = B.$$

□

Proof of Proposition 22.1.1. Suppose $B \subset [0,1]$ is closed and non-empty. Then B is in fact closed as a subset of $\mathbb{R}$.[1] B is obviously bounded, so it follows that B is compact by Heine-Borel.

Since the inclusion map $B \to \mathbb{R}$ is continuous, the extreme value theorem tells us that B has a maximal element, call it $b \in B$. b must equal 1, else Lemma 22.1.2 would contradict the maximality of b. Likewise, the minimal element in B must equal 0.

This shows that B^C must *not* contain 0 and 1. So now let us assume that $A \subset [0,1]$ is non-empty, and both closed and open; then the previous argument for $B = A$ shows A^C must *not* contain 0 and 1. But if A is open, then A^C is closed, so the argument for $B = A^C$ shows leads to a contradiction unless $A^C = \emptyset$. That $A^C = \emptyset$ means $A = [0,1]$, as desired. □

This proposition is powerful. For example, we have the following:

Corollary 22.1.3. Let X be a discrete topological space and fix elements $x, x' \in X$. Then there exists a path from x to x' if and only if $x = x'$.

Proof. Suppose $\gamma : [0,1] \to X$ is continuous, and that x is in the image of γ. because X has the discrete topology, the singleton set $\{x\}$ is both closed and open. (To see this, recall that every subset of X is open in the discrete topology. In particular, both $\{x\}$ and its complement are open.) Thus, the preimage $\gamma^{-1}(\{x\})$ is both a closed and open subset of $[0,1]$. By Lemma 22.1.1, the preimage must be either empty or all of $[0,1]$. Because we assumed x to be in the image,

$$\gamma^{-1}(\{x\}) = [0,1].$$

In particular, γ is a constant function, so $\gamma(0) = \gamma(1) = x$. □

Example 22.1.4. So, if X is a discrete topological space with two or more elements, X is not path-connected.

[1] To see this, note that $B^C = W \cap [0,1]$ for some $W \subset \mathbb{R}$ open, by definition of subspace topology. Then we can check that $\mathbb{R} \setminus B = W \bigcup (\mathbb{R} \setminus B)$, so that B is open in $\mathbb{R}$. In fact, if $I \subset X$ is closed, then $B \subset I$ is closed if and only if B is also closed as a subset of X.

22.2 Connectedness

So, path-connectedness was an intuitive notion: We'll say a space is path-connected if any two points can be connected by a path. Confusingly, the term "path-connected" is not the same as the term "connected" in our culture.

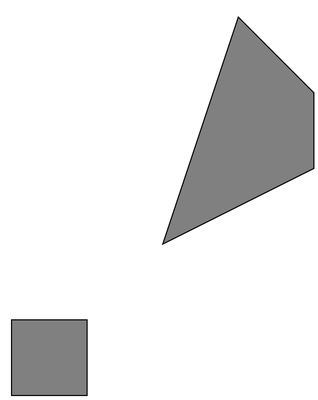
We now discuss a far less intuitive notion:

Definition 22.2.1. We say that a space X is *connected* if the following holds: If $A \subset X$ is both open and closed, then either $A = X$ or $A = \emptyset$.

Example 22.2.2. By Proposition 22.1.1, we know that $X = [0, 1]$ is a connected space.

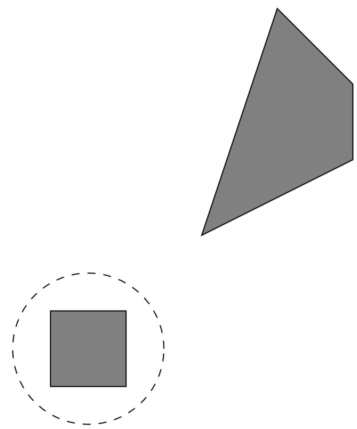
Example 22.2.3. Let X be a discrete topological space. If X has two or more elements, X is not connected.

Example 22.2.4. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

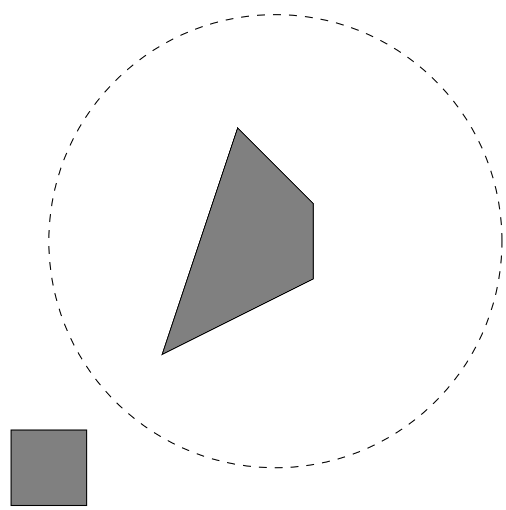


Let us label the lower-left component by A, and the upper-right component by B. I claim that both A and B are each both open and closed.

To see that A is open, simply observe that there is an open ball $W \subset \mathbb{R}^2$ for which $W \cap X = A$ (and then cite the definition of the subspace topology, which defines the topology on $X \subset RR^2$):



Because $B = A^C \subset X$, we conclude B is closed. To see B is open, likewise observe an open ball in X containing B but not A:



So B is open, meaning $A = B^C$ is closed. This shows $A \subset X$ is both open and closed, but $A \neq X$ and $A \neq \emptyset$.

Notice that all our examples connectedness/path-connectedness are the same. This is because of the following:

Proposition 22.2.5. If X is path-connected, then X is connected.

Proof. We will prove the contrapositive—that is, if X is not connected, then X is not path-connected.

Because X is not connected, there exists a subset $A \subset X$ which is non-empty, not all of X, but both open and closed.

So choose $x \in A$, and choose $x' \in A^C \subset X$. I claim there is no path from x to x'.

To see this, suppose we have a continuous map $\gamma : [0,1] \to X$ for which γ intersects A, we must have that $\gamma^{-1}(A)$ is non-empty. On the other hand, A is both open and closed, so $\gamma^{-1}(A)$ is both open and closed—this means $\gamma^{-1}(A) = [0,1]$ by Proposition 22.1.1.

That is, if $\gamma(t) \in A$ for some t, then $\gamma(t) \in A$ for every $t \in [0,1]$. In particular, if $x = \gamma(0)$, then $x' \neq \gamma(1)$. This proves the claim, and hence the proposition. □

Warning 22.2.6. There exist connected spaces that are not path-connected.

22.3 Stereographic projection

Stereographic projection is the function

$$p : S^n \setminus \{(0,\dots,0,1)\} \to \mathbb{R}^n, \qquad (x_1,\dots,x_{n+1}) \mapsto \frac{1}{1-x_{n+1}}(x_1,\dots,x_n).$$

Here is a description of p in words. For brevity, let us call the point $(0,\dots,0,1) \in S^n$ the *north pole* of S^n. Given a point $x \in S^n$ such that x is not the north pole, p sends x to the intersection of

- the line through x and the north pole, with
- the hyperplane $\{x_{n+1} = 0\}$, which one can identify with $\mathbb{R}^n$.

Informally, stereographic projection is the function obtained as follows: Given $x \in S^2$, we draw the line from the north pole through x. This line intersects a unique point of the plane $\{x_3 = 0\}$, and this point is $p(x)$. See Figure 22.1.

Note that the domain of stereographic projection is not all of S^n, but S^n minus a north pole. Notice also that p is a bijection; this gives us an informal way to think about S^n—it is obtained from $\mathbb{R}^n$ by "adding one point" that plays the role of the north pole of S^n.

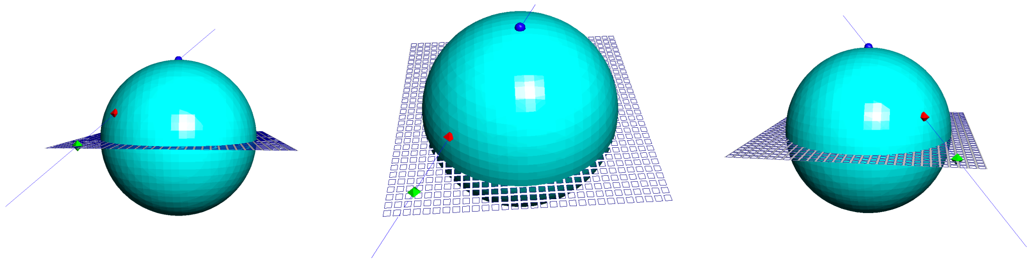


Figure 22.1: A depiction of stereographic projection for S^2. In blue is the north pole; the red is an element $x \in S^2$, and in green is the image of x under stereographic projection. Drawn also are S^2, the plane $\mathbb{R}^2$ (embedded as the subset where $x_3 = 0$) and the line connecting x to its image.

Remark 22.3.1. Here is one way to think about this: Imagine a nice smooth rubber ball. If you puncture the rubber ball in one place (say, with a needle), you can actually stretch out the entire rubber ball onto a flat surface. In fact, by stretching and stretching, you can cover the entire plane.

And, by adding this one point to $\mathbb{R}^n$, we obtain a compact topological space (the sphere). It is important here that we know how to *topologize* this set obtained by adjoining a point to $\mathbb{R}^n$. (There are ways to topologize this set that do not result in S^n, for example.) This process of adjoining one point to a space, to obtain a new, compact space, is called *one-point compactification*.

22.4 One-point compactification

Definition 22.4.1. Let X be a topological space. We are now going to create a new topological space X^+.

As a set, $X^+ = X \coprod \{*\}$. In other words, X^+ is the set obtained by adjoining a single point called $*$ to X.

The topology $\mathcal{T}_{X^+}$ is defined as follows: $U \subset X^+$ is open if either

1. $* \notin U$ and U is open in X, or

2. $* \in U$ and $U \cap X$ is the complement of a closed, compact subspace of X.

We call X^+ the *one-point compactification* of X.

Remark 22.4.2. Note that if X is Hausdorff, we may remove the adjective "closed" from the second condition above.

22.5 Basic properties/Exercises

Proposition 22.5.1. $\mathcal{T}_{X^+}$ is a topology on the set X^+.

(We need to prove that the collection of sets U satisfying 1. or 2. satisfies all the properties of a topology. You will want to use at some point that the empty set is a compact space.)

Proposition 22.5.2. X^+ is compact.

(We need to prove that every open cover of X^+ admits a finite subcover.)

Remark 22.5.3. This justifies the word "compactification."

Proposition 22.5.4. If X and Y are homeomorphic, so are X^+ and Y^+.

Proposition 22.5.5. If X is Hausdorff, so is X^+.

22.6 Examples/Exercises

Proposition 22.6.1. If X is compact, then X^+ is homeomorphic to the space $X \coprod \{*\}$ with the coproduct topology.

Proposition 22.6.2. If $X = \mathbb{R}^n$, then X^+ is homeomorphic to S^n. (This is one of your homework assignments.)

Solutions

Proof of Proposition 22.5.1. (i) We first show $\emptyset, X^+$ is in this topology. So let $U = \emptyset$. Then $* \notin U$, so we must check whether $\emptyset$ is open in X (by condition 1 of the definition of $\mathcal{T}_{X^+}$). It is, by definition of topological space (i.e., because X itself is a topological space). Now let $U = X^+$. Since $* \in U$, we must check whether $U \cap X$ is the complement of a closed, compact subspace of X (by condition 2 of the definition of $\mathcal{T}_{X^+}$). It is, because $U \cap X = X$ and X is the complement of $\emptyset$. (Note that $\emptyset$ is both closed and compact.)

(ii) Now let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an arbitrary collection where $U_\alpha \in \mathcal{T}_{X^+}$ for any $\alpha \in \mathcal{A}$. We must show that the union

$$U := \bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset X^+$$

is in $\mathcal{T}_{X^+}$.

Note that for any $\alpha \in \mathcal{A}$, we know that

$$U_\alpha \cap X$$

has a complement given by a closed subspace of X. (This is true regardless of whether U_α satisfies case 1. or in case 2. of the definition of $\mathcal{T}_{X^+}$.) Let us call this closed subspace K_α, and let us call the intersection

$$K := \bigcap_{\alpha \in \mathcal{A}} K_\alpha.$$

Note that the arbitrary intersection of closed subsets is closed, so $K \subset X$ is closed. Then by de Morgan's laws, we see that

$$X \cap U = X \cap \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) = (\bigcap_{\alpha \in \mathcal{A}} K_\alpha)^C = K^C.$$

where the complement is taken inside X. Now, if $* \notin U$, then we have shown that U^C is closed, so by condition 1. of the definition of $\mathcal{T}^+$, we see that $U \subset X^+$ is indeed in $\mathcal{T}_{X^+}$.

On the other hand, if $* \in U$, then for some $\alpha \in \mathcal{A}$, we see that $* \in U_\alpha$. In particular, K_α is not only closed, but also compact. Thus $K \subset K_\alpha$ is a closed subspace of a compact K_α, meaning K itself is compact. This shows that $U \cap X = K^C$ is the complement of a compact, closed subspace of X, so U is open by condition 2. of the definition of $\mathcal{T}_{X^+}$.

(iii) Now we must show that a finite intersection of elements in $\mathcal{T}_{X^+}$ is in $\mathcal{T}_{X^+}$.

So fix $U_1, \ldots, U_n$, a finite collection of elements in $\mathcal{T}_{X^+}$. For each i, let $K_i = (U_i \cap X)^C$. Note that K_i is closed, and is compact if $* \in U_i$. We let

$$U = U_1 \cap \ldots \cap U_n \subset X^+$$

and

$$K = K_1 \cup \ldots \cup K_n \subset X.$$

Note that by de Morgan's laws, we again have

$$U \cap X = K^C \subset X$$

(where the complement is again taken inside X).

If $* \notin U$, then $U = K^C$. Being a complement of a closed subset in X, we see that $U \subset X$ is open in X, so $U \in \mathcal{T}_{X^+}$ by condition 1. of the definition.

If $* \in U$, then $* \in U_i$ for every i, so by condition 2, each K_i is not only closed but also compact. Lemma: The finite union of compact subspaces is compact. (Proof: Given an open cover of K, note that the open cover determines a finite subcover of each K_i. Taking the union of these finite subcovers, we have a finite union of finite collections; hence the resulting union is a finite open cover of K itself.) Thus K itself is compact. By condition 2, U is in $\mathcal{T}_{X^+}$. □

Proof of Proposition 22.5.2. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X^+. By definition of cover, there is some $\alpha_0 \in \mathcal{A}$ such that $* \in U_{\alpha_0}$. So by condition 2 of the definition of $\mathcal{T}_{X^+}$, we know

$$X^+ = U_{\alpha_0} \cup K$$

where K is a compact, closed subspace of X and $K \cap U_{\alpha_0} = \emptyset$.

Before we go any further, let us point out that $X \subset X^+$ is an open subset by condition 1 of the definition of $\mathcal{T}^+$. Thus the subspace topology of $K \subset X^+$ is equal to the subspace topology of $K \subset X$.

Invoking the definition of open cover, and by definition of subspace topology (for $K \subset X$), we know that the collection

$$\{U_\alpha \cap K\}_{\alpha in \mathcal{A}}$$

is an open cover of K. Since K is compact, we can choose some finite collection $\alpha_1, \ldots, \alpha_n$ so that $\{U_{\alpha_1} \cap K, \ldots, U_{\alpha_n} \cap K\}$ is an open cover of K. In particular,

$$U_{\alpha_0} \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$$

is an open cover of X^+ itself. This exhibits a finite subcover of the original open cover, and we are finished. □

Proof of Proposition 22.6.1. We must show that $W \subset X^+$ is open if and only if $W \cap X$ and $W \cap \{*\}$ is open.

To see the latter claim, we must prove that the one-element set

$$U = \{*\} \subset X^+$$

is open. This is because $U \cap X = \emptyset = X^C$, where the complement is taken in X. But X is closed (as a subset of itself), and is compact by hypothesis, so by condition 2, U is open.

On the other hand, $W \cap X$ is always open for a one-point compactification—this is obvious if $* \notin W$ by condition 1, and if $* \in W$, then $W \cap X$ is a complement of a (compact and) closed subset of X by condition 2, hence by definition of closedness, $W \cap X$ is open in X.

This completes the proof. □

Proof of Proposition 22.6.2. Omitted, as it is entirely analogous to the solutions to homework. □

Proof of Proposition 22.5.4. Given a homeomorphism $f : X \to Y$, define a function

$$g : X^+ \to Y^+, \qquad x \mapsto \begin{cases} *_Y & x = *_X \\ f(x) & x \in X. \end{cases}$$

Here, $*_Y \in Y^+$ represents the "extra point" in the one-point-compactification of Y, and likewise for $*_X \in X^+$.

Clearly g is a bijection because f is. Let us show that $U \subset X^+$ is open if and only if $g(U) \subset Y^+$ is open.

1. If $*_X \notin U$, then $*_Y \notin g(U)$. But because f is a homeomorphism, $g(U) = f(U)$ is open if and only if $U \cap X = U$ is open.

2. If $*_X \in U$, then $*_Y \in g(U)$. This means that $U \cap X = K^C$ (where the complement is taken in X) for some compact, closed $K \subset X$. But because f is a homeomorphism, $K \subset X$ is compact and closed if and only if $f(K) \subset Y$

is also compact and closed. Thus $f(U) \cap Y$ is the complement of a closed, compact subspace of Y if and only if $U \cap X$ is the complement of a closed, compact subspace of X. This completes the proof. $\square$