# Lecture 23

# Density, Interiors, Closures, Neighborhoods

## 23.1 Closure

**Definition 23.1.1.** Fix a topological space $X$ and let $B \subset X$ be a subset.[1] Let

$$\mathcal{K}_B$$

be the collection of all closed subsets of $X$ containing $B$.[2] Then the *closure* of $B$ is defined to be

$$\overline{B} := \bigcap_{K \in \mathcal{K}_B} K.$$

In words, the closure of $B$ is the set obtained by intersecting every closed subset containing $B$.

**Remark 23.1.2.** Note that $B$ is always a subset of $\overline{B}$.

**Remark 23.1.3.** Note that $\overline{B}$ is a closed subset of $X$. This is because the intersection of closed subsets is always closed.

**Remark 23.1.4.** If $B \subset X$ is closed, then $\overline{B} = B$. To see this, note that $B$ is an element of $\mathcal{K}$ because $B$ is closed. Hence

$$\bigcap_{K \in \mathcal{K}} K = B \cap \left( \bigcap_{K \in \mathcal{K}, K \neq B} K \right).$$

[1]It could be any kind of subset: open, closed, neither!
[2]Note that $X$ is an element of $\mathcal{K}_B$.

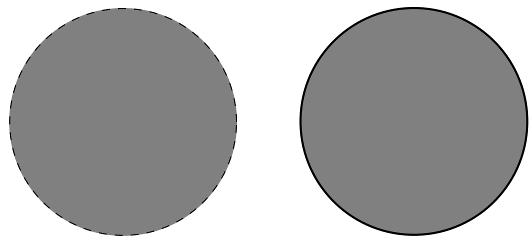

Figure 23.1: An open ball on the right; its closure (a closed ball) on the left.

But this righthand side is a subset of $B$ because it is obtained by intersecting $B$ with some other set. In particular,

$$\overline{B} \subset B.$$

Because $B \subset \overline{B}$ (for any kind of $B$), we conclude that $B = \overline{B}$.

**Example 23.1.5.** If $B = \emptyset$, then $\overline{B} = \emptyset$. If $B = X$, then $\overline{B} = X$.

**Exercise 23.1.6.** Let $X = \mathbb{R}^n$ (with the standard topology). Let $B = \text{Ball}(0, r)$ be the open ball of radius $r$. Show that the closure of $B$ is the closed ball of radius $r$; that is,

$$\overline{B} = \{x \in \mathbb{R}^n \text{ such that } d(x, 0) \leq r \ .\}$$

*Proof.* You showed in your homework that if $K \subset X$ is closed and if $x_1, \ldots$ is a sequence in $K$ converging to some $x \in X$, then $x$ is in fact an element of $K$.

Choose a point $x$ of distance $r$ from the origin. And choose also an increasing sequence of positive real numbers $t_1, t_2, \ldots$ converging to 1.[3] Then the sequence

$$x_i = t_i x$$

is a sequence in $B$ converging to $x$. If $K \supset B$, then the $x_i$ define a sequence in $K$; moreover, if $K$ is closed, the limit $x$ is in $K$. Thus $x \in K$ for any closed subset containing $B$. In particular, $x$ is in the intersection of all such $K$. Thus $x \in \overline{B}$. This shows that the closed ball of radius $r$ is contained in $\overline{B}$.

[3]For example, you could take $t_i = i/(i+1)$.

On the other hand, consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $d(0, -)$; that is, the "distance to the origin" function. We see that $f^{-1}([0, r])$ is equal to the closed ball of radius $r$—in particular, this closed ball is a closed subset of $\mathbb{R}^n$, and it obviously contains $\text{Ball}(0, r)$. This shows that $\overline{B}$ is a subset of the closed ball of radius $r$ (because $\overline{B}$ can be expressed as the intersection of this closed ball with other sets). We are finished. □

**Exercise 23.1.7.** Suppose $f : X \to Y$ is a continuous function, and let $B \subset X$ be a subset. Show that

$$f(\overline{B}) \subset \overline{f(B)}.$$

In English: The image of the closure of $B$ is contained in the closure of the image of $B$.

*Proof.* Let $\mathcal{C}$ be the collection of closed subsets of $Y$ containing $f(B)$. Then

$$f^{-1}(\overline{f(B)}) = f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right)$$

by definition of closure. We further have:

$$f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right) = \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

Now, because $f$ is continuous, we know that $f^{-1}(C)$ is closed for every $C \in \mathcal{C}$. Moreover, because $f(B) \subset C$, we see that $B \subset f^{-1}(C)$. We conclude that for every $C \in \mathcal{C}$, $f^{-1}(C) \in \mathcal{K}$. Thus

$$\bigcap_{K \in \mathcal{K}} K \subset \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

The lefthand side is the definition of $\overline{B}$. The righthand side is $f^{-1}(\overline{f(B)})$. We are finished. □

**Remark 23.1.8.** It is not always true that $f(\overline{B})$ is equal to $\overline{f(B)}$. For example, let $B = X = \text{Ball}(0, r)$, and let $f : X \to \mathbb{R}^2$ be the inclusion. Then $f(\overline{B}) = X$, while $\overline{f(B)}$ is the closed ball of radius $r$.

**Exercise 23.1.9.** Find an example of a continuous function $p : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{\{x \text{ such that } p(x) < t\}},$$

does *not* equal

$$\{x \text{ such that } p(x) \leq t\}.$$

**Example 23.1.10.** Let $B \subset \mathbb{R}^2$ be the following subset:

$$B = \{(x_1, x_2) \text{ such that } x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\} \subset \mathbb{R}^2.$$

This is not a closed subset of $\mathbb{R}^2$. I claim

$$\overline{B} = B \bigcup \{(x_1, x_2) \text{ such that } x_1 = 0 \text{ and } x_2 \in [-1, 1]\}.$$

That is, $\overline{B}$ is equal to the so-called topologist's sine curve.

Let us call the righthand side $S$ for the time being. First, I claim that $S \subset \overline{B}$. Indeed, fix some point $(0, T) \in S \setminus B$. Then there is an unbounded, increasing sequence of real numbers $t_1, t_2, \ldots$ for which $\sin(t_i) = T$; let $s_i = 1/t_i$. Then the sequence of points

$$x_i = (s_i, \sin(1/s_i)) = (s_i, T)$$

converges to $(0, T)$, while each $x_i$ is an element of $B$. In particular, $(0, T)$ is contained in any closed subset containing $B$. This shows $S \subset \overline{B}$.

To complete the proof, it suffices to show that $S$ is closed. For this, because $\mathbb{R}^2$ is a metric space, it suffices to show that any convergent sequence contained in $S$ has a limit contained in $S$. So let $x_1, x_2, \ldots$ be a sequence in $S$.

Suppose that the limit $x \in \mathbb{R}^2$ has the property that the 1st coordinate is non-zero. There is a unique point in $S$ with a given non-zero first coordinate $t$, namely $(t, \sin(1/t))$. Moreover, because the function $t \mapsto \sin(t/1)$ is continuous, if $t_i = \pi_1(x_i)$ converges to $t$, we know that $(t_i, \sin(1/t_i))$ converges to $(t, \sin(1/t))$. So the limit is in $S$.

If on the other hand the first coordinate of $x$ is equal to zero, let us examine the second coordinates $\pi_2(x_1), \ldots$. By continuity of $\pi_2$, the sequence $\pi_2(x_1), \pi_2(x_2), \ldots$ converges to some $T$; because each $x_i$ has a second coordinate in $[-1, 1]$, and because $[-1, 1] \subset RR$ is closed, we conclude that the limit $T$ is also contained in $[-1, 1]$. Hence the limit of the sequence $x_1, \ldots,$ is the point $(0, T)$, and $(0, T) \in S$.

Because any sequence in $S$ with a limit in $\mathbb{R}^2$ has limit in $S$, $S$ is closed.

## 23.2 Interiors

**Definition 23.2.1.** Let $X$ be a topological space and fix $B \subset X$. Let $\mathfrak{U}_B$ denote the collection of open subsets of $X$ that are contained in $B$. Then the *interior* of $B$ is defined to be the union

$$int(B) = \bigcup_{U \in \mathfrak{U}_B} U.$$

**Remark 23.2.2.** For any $B$, we have that $int(B) \subset B$. Moreover, $int(B)$ is an open subset of both $B$ and of $X$.

**Remark 23.2.3.** If $B$ is open, then $int(B) = B$. This is because $B \in \mathfrak{U}_B$, so

$$int(B) = \bigcup_{U \in \mathfrak{U}_B} U = B \cup \left( \bigcup_{U \neq B, U \in \mathfrak{U}_B} U \right)$$

meaning $int(B)$ contains $B$ (because $int(B)$ is a union of $B$ with possibly other sets). Thus we have that $int(B) \subset B \subset int(B)$, meaning $int(B) = B$.

**Example 23.2.4.** We have that $int(\emptyset) = \emptyset$ and $int(X) = X$.

**Example 23.2.5.** Let $X = \mathbb{R}^n$ and let $B$ be the closed ball of radius $r$. Then $int(B) = \text{Ball}(0, r)$ is the open ball of radius $r$.

To see this, we note that $\text{Ball}(0, r)$ is open and contained in $B$, so $\text{Ball}(0, r) \subset int(B)$ by definition of interior. Because $int(B) \subset B$, it suffices to show that no other point of $B$ (i.e., no point in $B \setminus \text{Ball}(0, r)$) is contained in the interior of $B$.

So fix $y \in B \setminus \text{Ball}(0, r)$, meaning $y$ is a point of exactly distance $r$ away from the origin. It suffices to show that there is no open ball containing $y$ and contained in $B$; for then there is no $U \in \mathfrak{U}$ for which $y \in U$.

Well, for any $\delta > 0$, $\text{Ball}(y, \delta) \subset \mathbb{R}^2$ contains some point of distance $> r$ from the origin. So $\text{Ball}(y, \delta)$ is never contained in $B$. This completes the proof.

## 23.3 Neighborhoods

**Definition 23.3.1.** Let $X$ be a space and let $x$ an element of $X$. A subset $A \subset X$ is called a *neighborhood* of $x$ if there exists some open $U \subset X$ with $x \in U$ for which $U \subset A$.

When $A$ is also open, we say $A$ is an *open neighborhood* of $x$.

**Example 23.3.2.** Let $U$ be an open subset containing $x$. Then the closure $\overline{U}$ is a neighborhood of $x$.

**Remark 23.3.3.** In topology, we use the word "neighborhood" usually when we're being lazy. "Neighborhood of $x$" is shorter than saying "subset with wiggle room about $x$."

Note that a neighborhood of $x$ need not be an open subset.

## 23.4 Density

**Definition 23.4.1.** Let $X$ be a topological space and fix a subset $B \subset X$. We say that $B$ is *dense* in $X$ if $\overline{B} = X$.

Prove the following:

**Proposition 23.4.2.** Fix $B \subset X$. The following are equivalent:

1. $B$ is dense in $X$.
2. For every non-empty open $U \subset X$, $U \cap B \neq \emptyset$.
3. For every $x \in X$, and every neighborhood $A$ of $x$ in $X$, we have that $A \cap B \neq \emptyset$.
4. For every $x \in X$, and every open neighborhood $A$ of $x$ in $X$, we have that $A \cap B \neq \emptyset$.

**Proposition 23.4.3.** $\mathbb{Q} \subset \mathbb{R}$ is dense.

**Proposition 23.4.4.** $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

**Exercise 23.4.5.** For each of the following examples of subsets of $\mathbb{R}^2$, identify the closure, the interior, and the boundary. Which of these is dense?

1. $B = \{(x_1, x_2) \text{ such that } x_1 \neq 0\ \}$.
2. $B = \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} (a-1, a+1) \times (b-1, b+1)$.
3. $B = \{(x_1, x_2) \text{ such that at least one of the coordinates is rational}\}$.

## Solutions to Lecture 23 Propositions

*Proof of Proposition 23.4.2.* There is a mistake in this problem: Condition 2 should say that for every *non-empty* $U \subset X$, we have $U \cap B \neq \emptyset$.

$1 \implies 2$. Proof by contrapositive. Suppose that there is some non-empty open $U \subset X$ such that $U \cap B = \emptyset$. Then $U^C$ is closed while $U^C \supset B$, so the closure of $B$ is contained in $U^C$ by definition of closure. In particular, $\overline{B}$ does not contain $U$, so could not equal all of $X$.

$2 \implies 4$. This is obvious, as if $A$ is an open neighborhood of $x$, then $A$ is a non-empty open subset of $X$.

$4 \implies 3$. Given $A$ a neighborhood of $x$, let $U \subset A$ be the open subset containing $x$ (guaranteed by the definition of neighborhood). Then $U \cap B \neq \emptyset$ by 4, so $A \cap B \supset U \cap B \neq \emptyset$.

$3 \implies 1$. Clearly $\overline{B} \subset X$ always, so we must show that $X \subset \overline{B}$. Let $K \subset X$ be a closed subset containing $B$. Then $K^C$ is open. If $K^C$ is non-empty, choose $x \in K^C$, and note that $K^C$ is a neighborhood of $x$. Thus by 3, $K^C \cap B \neq \emptyset$; this contradicts the fact that $B \subset K$. □

*Proof of Proposition 23.4.3.* Let $x \in \mathbb{R}$ be a real number, and for every integer $n \geq 1$, let $x_n$ be any rational number in the interval $(x - 1/n, x + 1/n)$. Then the sequence $x_n$ converges to $x$. By the sequence criterion for closure, we thus see that any real number is in the closure of $\mathbb{Q}$. □

*Proof of Proposition 23.4.4.* Same exact proof, except choose each $x_n$ to be any *irrational* number in the interval $(x - 1/n, x + 1/n)$. □

*Proof of Proposition* **??**. $int(B)$ is open in $X$ because it is a union of open sets. (And unions of open sets are always open by definition of topology.) It is open in $B$ because

$$int(B) \cap B = int(B),$$

and by definition of subspace topology, a subset of $B$ is open if and only if it is an intersection of $B$ with an open subset (like $int(B)$) of $X$.

Finally, $int(B) \subset B$ because $int(B)$ is a union of subsets of $B$. □

*Proof of Proposition* **??**. If $B$ is open, then obviously $B \in \mathfrak{U}$, while $U \in \mathfrak{U} \implies U \subset B$, so $\bigcup_{U \in \mathfrak{U}} U \subset B$ while $B \subset \bigcup_{U \in \mathfrak{U}} U$. Hence $B = int(B)$.

On the other hand, if $B = int(B)$, then $B$ is a union $\bigcup_{U \in \mathfrak{U}} U$ of open subsets of $X$; hence $B$ is an open subset of $X$. □