Extra Credit Exam I Quizzes (Deadline April 1, 10:30 AM)

For extra credit, you may take up to four extra credit quizzes. The four quizzes will be on the topics of:

1. Implicit Differentiation,
2. Minima and Maxima,
3. Related Rates, and
4. Taylor Polynomials.

I **strongly encourage** you to study and take all four.

Points. Each quiz will be worth at most 10 points, and these points will be added directly to your Exam I score. So for example, if you scored 60 points on the exam (since the exam is out of 144 points, 60 points is roughly 42%), you have a chance to change your grade to a maximum of 100 points (which would be roughly 70%).

Deadline. Whichever quizzes you choose to take, you must take them by April 1, 10:30 AM (to complete them by April 1, 10:40 AM).

Logistics. To take a quiz, you come to my office hours. In my office, I will give you a quiz, and you must complete it within 10 minutes. **If you cannot come to my office hours**, email me, and we will schedule a time for you to take the quiz. You do not need to take all four quizzes in one sitting—you may come to my office four separate times, for example, to take the four quizzes. **You may not repeat** a quiz topic; so if you take four quizzes, each quiz must be on a different topic.

Prompt. In the pages that follow, I have given three examples of the kinds of problems that will be on these extra credit quizzes. I have given solutions to each example. The quiz will consist of a problem very similar to the ones presented in this PDF file.

2 Implicit Differentiation Problems

(I) Using implicit differentiation, find the slope of the tangent line, at a point (x, y), to the shape cut out by the equation

$$xe^y + ye^x = 0.$$

(II) Consider the shape

$$3yx^2 - y^3 - x^3 = 0.$$

Using implicit differentiation, find the slope of the tangent line to this shape at a point (x, y).

(III) An ant is walking along a curve defined by the equation

$$\cos(xy) + y\cos(x) = x\sin(y).$$

What is the slope of the ant's path if the ant is at the point (x, y)?

2.1 Solutions

Remember, in implicit differentiation, we pretend y is a function of x, so we can pretend $y = f(x)$. When we differentiation terms involving y, this means we'll make frequent use of product and chain rules. For example,

$$(xy)' = (xf(x))' = x'f(x) + x \cdot f'(x) = x'y + xy' = y + xy'$$

Likewise,

$$(y^2)' = (f(x)^2)' = 2f(x)f'(x) = 2y \cdot y'.$$

Note that we are substituting back y' for $f'(x)$.

Often, to save time, we skip the step of pretending $y = f(x)$ and just quickly write $(y^2)' = 2y \cdot y'$. It is up to you whether you want to do this "time-saving" method or to carefully substitute $f(x)$ for y and apply the differentiation rules carefully.

(I) We have

$$\begin{aligned}
(xe^y + ye^x)' &= (xe^y)' + (ye^x)' && (31)\\
&= (x)'e^y + x(e^y)' + (y)'e^x + y(e^x)' && (32)\\
&= e^y + xe^y y' + y'e^x + ye^x && (33)\\
&= y'(xe^y + e^x) + e^y + ye^x. && (34)
\end{aligned}$$

Because $xe^y + ye^x = 0$, we know that $(xe^y + ye^x)' = 0' = 0$. So we obtain

$$y' = \frac{-e^y - ye^x}{xe^y + e^x}.$$

(II)

$$\begin{aligned} 0 = 0' = (3yx^2 - y^3 - x^3)' &= (3yx^2)' - (y^3)' - (x^3)' && (35) \\ &= 3y'(x^2) + 3y \cdot 2x - 3y^2y' - 3x^2 && (36) \\ &= 3y'\left(x^2 - y^2\right) + 6xy - 3x^2. && (37) \end{aligned}$$

Hence

$$y' = \frac{2xy - x^2}{y^2 - x^2}.$$

(III) Since the curve is defined by

$$\cos(xy) + y\cos(x) = x\sin(y)$$

we also know

$$0 = \cos(xy) + y\cos(x) - x\sin(y). \qquad (38)$$

Taking derivatives as usual, we find

$$\begin{aligned} (\cos(xy) + y\cos(x) - x\sin(y))' &= (\cos(xy))' + (y\cos(x))' - (x\sin(y))' && (39) \\ &= -\sin(xy)\cdot(xy)' \\ &\quad +(y)'\cos(x) + y(\cos(x))' \\ &\quad -(x)'\sin(y) - x(\sin(y))' && (40) \\ &= -\sin(xy)\cdot(x\cdot y' + y) \\ &\quad +y'\cos(x) + y\cdot(-\sin(x)) \\ &\quad -\sin(y) - x\cdot\cos(y)y' && (41) \\ &= y'(-x\sin(xy) + \cos(x) - x\cos(y)) \\ &\quad +(-y\sin(xy) - y\sin(x) - \sin(y)). && (42) \end{aligned}$$

Because of (38), we know that $0 = 0' = (\cos(xy) + y\cos(x) - x\sin(y))''$, so we from (42) we conclude

$$0 = y'(-x\sin(xy) + \cos(x) - x\cos(y)) + (-y\sin(xy) - y\sin(x) - \sin(y)).$$

Isolating y', we find

$$y' = \frac{y\sin(xy) + y\sin(x) + \sin(y)}{-x\sin(xy) + \cos(x) - x\cos(y)}.$$

3 Min/Max Problems

These Min/Max problems are a little tougher than the ones from lab, because you have to *find* the function that you're optimizing—the function is not just given to you. Moreover, the function you want to optimize often depends on more than one input variable (for example, the area depends on both length *and* width), so you have to find a relationship that allows you to express the function using only one input variable.

(I) Bena is making a rectangular shape out of piping. One side of this rectangular shape must made of (pricier) copper, worth 2 dollars per foot of piping. The other three sides must be made of PVC (a type of plastic), worth about 50 cents per foot of piping. The piping must enclose a region with 30 square feet.

  (a) How long should the copper piping be to minimize cost?

  (b) What is this minimized cost?

(II) Consider the set of points (x, y) satisfying $y = 3x + 2$. (This set of points forms a line, as you know.)

  (a) Which point on this line is closest to the origin of the plane?

  (b) What is the distance of that point from the origin?

(III) Melody is making a rectangle out of 40 centimeters of string.

  (a) What are the length and width of the rectangle of maximal area she can make?

  (b) What is the maximal area she can enclose in her rectangle?

3.1 Solutions

(I) Let's set up the problem. Let's call w the width of the rectangular region, and l the length. Then we know that

$$\text{Area} = wl. \tag{43}$$

How much is the cost? Well, there are four sides to a rectangle. Let's say that a side of length w is made of copper[6], while the other three sides are made of PVC. Then the total cost is given by

$$\text{Cost} = (2 \text{ dollars/ft}) \times (w \text{ ft of copper}) + (0.5 \text{ dollars/ft}) \times (w + 2l \text{ feet of PVC}).$$

Or, getting rid of the words on the right, we have

$$\text{Cost} = 2w + 0.5 \times (w + 2l). \tag{44}$$

Part (a) is asking us for the length of copper, which we chose to be w. So l seems like a bit of a distraction. Let's remember the equation for area (43), and note that we *need* the area to be 30. In other words, (43) and the problem's constraints tell us

$$30 = wl.$$

So whatever l is, we know

$$l = 30/w. \tag{45}$$

So—because (a) is asking us for the length w of copper—let's eliminate the appearance of the distracting l by substiting (45) into (44), to find:

$$\text{Cost} = 2w + 0.5 \times (w + 2\frac{30}{w}).$$

Simplifying, we find

$$\text{Cost}(w) = w.5w + \frac{30}{w}. \tag{46}$$

Now, we want to *minimize* cost, so at this point we see that we are looking for the absolute minimum of the function in (46). To find this minimum, let's look for critical points. To look for critical points, we must find out where $\text{Cost}'(w) = 0$. Well,

$$\text{Cost}'(w) = 2.5 - \frac{30}{w^2}.$$

[6] If you wanted, you could instead call l the length of the copper; then the roles of w and l will be swapped from hereon.

If we want $\text{Cost}'(w) = 0$, then we find

$$2.5 = \frac{30}{w^2}$$

meaning we want

$$w^2 = \frac{30}{2.5} = 12.$$

So the only critical points are at $w = \sqrt{12}$ and $w = -\sqrt{12}$. Note that because we are looking for a physical length of pipe, w cannot be a negative number, so we end up with only one

$$\text{critical point given by } w = \sqrt{12}. \tag{47}$$

We must now check whether this is indeed a minimum. (After all, what if the cost function doesn't even have minima?) For this, let's check the concavity of the cost function. We find

$$\text{Cost}''(w) = 0 + \frac{2 \cdot 30}{w^3} = \frac{60}{w^3}.$$

This is *always positive* so long as w is positive (which w must be, being the length of something). In other words, Cost is always concave up as a function of w. Not only can we now conclude that our critical point from (47) is a local minimum, we conclude it is an *absolute* minimum because our function $\text{Cost}(w)$ is always concave up! Thus $w = \sqrt{12}$ is an absolute minimum.

So we can confidently say: **Answer to (a):** The length of the copper side must be $\sqrt{12}$ feet.

To find the answer to (b), we utilize the version of the Cost function that's easiest to evaluate using w—the one in (46). We find

$$\text{Cost}(\sqrt{12}) = 2.5\sqrt{12} + \frac{30}{\sqrt{12}} \tag{48}$$

$$= 2.5\sqrt{12} + \frac{30\sqrt{12}}{12} \tag{49}$$

$$= 2.5\sqrt{12} + \frac{5}{2}\sqrt{12} \tag{50}$$

$$= 5\sqrt{12}. \tag{51}$$

So **the answer to (b)** is $5\sqrt{12}$ dollars. In real life, we would probably round this to two decimal places.

If you wanted a decimal answer, you'd probably use a calculator to find that this is about 17 dollars and 32 cents.

Remark 3.1 (The actual work you can show on an exam)**.** If you needed to show your work on an exam, it will probably suffice to write something like the following:

$$\begin{aligned} 30 &= wl && (52)\\ l &= 30/w && (53)\\ \text{Cost} &= 2w + 0.5 \times (w + 2l) && (54)\\ &= 2w + 0.5 \times (w + 2\frac{30}{w}) && (55)\\ &= 2.5w + \frac{30}{w}. && (56)\\ \text{Cost}' &= 2.5 - \frac{30}{w^2}. && (57)\\ & && (58) \end{aligned}$$

$$\begin{aligned} \text{Cost}'(w) = 0 &\implies 2.5 = \frac{30}{w^2} && (59)\\ &\implies w^2 = 30/2.5 = 60/5 = 12. && (60)\\ \text{so } w = \sqrt{12} &\text{ (negative value of } w \text{ doesn't make sense).} && (61) \end{aligned}$$

$$\begin{aligned} \text{Cost}''(w) &= \frac{60}{w^3} && (62)\\ &\text{positive for all } w\text{, so concave up, so crit pt is absolute maximum} && (63)\\ & && (64) \end{aligned}$$

$$\text{(a): } \sqrt{12} \text{ feet of copper}$$

$$\text{Cost}(\sqrt{12}) = 2.5\sqrt{12} + \frac{30}{\sqrt{12}} \tag{65}$$

$$= 2.5\sqrt{12} + \frac{30\sqrt{12}}{12} \tag{66}$$

$$= (2.5 + 2.5)\sqrt{12} \tag{67}$$

$$= 5\sqrt{12}. \tag{68}$$

$$\text{(b): } 5\sqrt{12} \text{ dollars}$$

Remark 3.2. You could set up the problem in other ways. For example, you could have gotten rid of w, and then solved for the minimizing l. You'd eventually have to find the minimizing w again using the equation $30 = wl$, but it would still be correct.

Instead of using w and l, the dimensions of the rectangle, you could have also used variables like C and P to represent the length of copper and PVC pipe, respectively. Then the two main equations to set you up would be $30 = CP$ and $\text{Cost} = 2C + 0.5C + 0.5(P - C)/2 + 0.5(P - C)/2$; note that in the cost equation, I am using that there are four sides to a rectangle (so there are four terms being added) of length C and of length $(P - C)/2$. This way is a bit more complicated, as you can see, to set up; but the end answer won't change for $C = w$.

(II) Set-up: Distance to the origin is given by the function

$$\sqrt{x^2 + y^2}. \tag{69}$$

y and x must satisfy a relation called $y = 3x + 2$, so we substitute this relation into (69) to obtain

$$\sqrt{x^2 + (3x + 2)^2}.$$

We must minimize this function. Now, note that to minimize the square root of BLAH, it suffices to minimize BLAH. (The smaller the number, the smaller the square root.) So let's minimize instead

$$x^2 + (3x + 2)^2.$$

Simplifying, we must minimize the expression

$$f(x) = 10x^2 + 12x + 4.$$

(The letter f was chosen arbitrarily.) To minimize this expression, let's take the derivative to find critical points:

$$f'(x) = 20x + 12.$$

So the only critical point is at $x = -12/20 = -3/5$. Let's check to see if this is a minimum by finding concavity:

$$f''(x) = 20.$$

So yes, the entire function d is concave up, so the lone critical point must be an absolute minimum.

So the x-coordinate $-3/5$ minimizes distance to the origin. Using (69), we can recover the y-coordinate of the point we seek:

$$y = 3(-3/5) + 2 = 1/5.$$

Answer to (a): The minimizing point has coordinates $(-3/5, 1/5)$

Answer to (b): The minimizing distance is given by

$$\sqrt{x^2 + y^2} = \sqrt{(-3/5)^2 + (1/5)^2} = \sqrt{10/25} = \frac{\sqrt{10}}{5} = \frac{1}{5}\sqrt{10}.$$

(Any of the last three expressions is acceptable; the last two are preferred.)

(III) We have 40 cm of string; so if our rectangle has width w and length l, we know

$$40 = \text{perimeter of rectangle} 2w + 2l. \tag{70}$$

And we want to maximize area:

$$\text{Area} = wl. \tag{71}$$

Well, let's plug in the relationship given by (70) into the area function to obtain:

$$\text{Area} = w \cdot (20 - w). \tag{72}$$

(To spell this out: I saw that $2l = 40 - 2w$, so I saw that $l = 20 - w$. I substituted l for $20 - w$ in the area function.) Hence

$$\text{Area} = 20w - w^2.$$

Let's find the critical points of this function of w. The derivative is given by

$$\text{Area}' = 20 - 2w,$$

so there is a unique critical point given by $w = 10$. To check the concavity, we compute:

$$\text{Area}'' = -2.$$

The area function is hence always concave down (as a function of w), so the lone critical point we found must be not only a local maximum, but a global one!

Answer to (a): $w = 10$. By the constraint equation (70), we also see that $l = 10$. So this rectangle is in fact a square!

Answer to (b): Area is given by $wl = (10)(10) = 100$. Hence the maximal area she can enclose is 100 centimeters squared.

4 Related Rates Problems

(I) It is night, and a ball of light is falling. The height of the ball of light is given by a function $H(t)$ where t is in seconds, and $H(t)$ is in meters. 9 horizontal meters away is a pole of height 13 meters. The pole (because of the light from the ball of light) casts a shadow of length $s(t)$.

(a) Express $s(t)$ in terms of $H(t)$ and $H'(t)$.

(b) If $H(t) = 100 - 4.9t^2$, how fast is the shadow growing at $t = 2$?

(II) A spherical dome of radius R meters is being flooded with water. Later in this course, you will see that if the water level inside the dome has height h, then there are

$$\pi h(R^2 - \frac{h^2}{3})$$

cubic meters of water in the dome.

(a) If the water's volume is increasing at 10 cubic meters per second, and if the water level is 1 meter, how quickly is the water level changing? Your answer should be in meters per second.

(b) If the water's volume is still increasing at 10 cubic meters per second, and if the water level is 2 meters, how quickly is the water level changing?

(c) As the water level increases, is the *rate of change* of the water level increasing or decreasing?

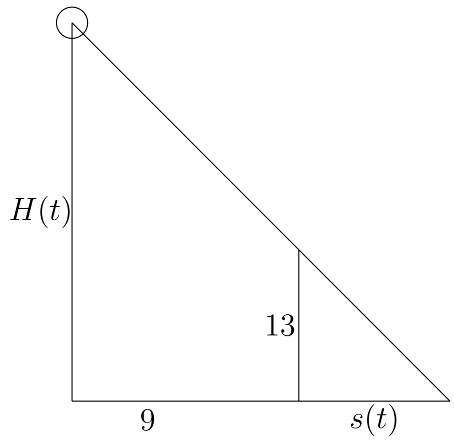
(III) When an earthquake occurs, certain surface waves called Raleigh waves emanate from the epicenter of the earthquake at 3.4 kilometers per second. Remember that the area of a circle of radius r is given by πr^2.

(a) After 10 seconds, how many square kilometers of land have been affected by the Raleigh waves?

(b) When the Raleigh wave has traveled a kilometer, how quickly is the area of affected land changing? Give your answers in kilometers squared per second.

4.1 Solutions

(I) Let's draw a picture:



The picture depicts in the upper-left the ball of light at height $H(t)$. 9 meters away is the pole of height 13 meters, and $s(t)$ is the shadow cast by the pole.

Using similar triangles, we see that

$$\frac{13}{s(t)} = \frac{H(t)}{9 + s(t)}.$$

We manipulate this equation (for example) as follows:

$$\frac{13}{H(t)}(9 + s(t)) = s(t) \tag{73}$$

$$\frac{13 \cdot 9}{H(t)} + s(t)\frac{13}{H(t)} = s(t) \tag{74}$$

$$\frac{13 \cdot 9}{H(t)} = s(t) - s(t)\frac{13}{H(t)} \tag{75}$$

$$\frac{13 \cdot 9}{H(t)} = s(t)[1 - \frac{13}{H(t)}] \tag{76}$$

$$\frac{13 \cdot 9}{H(t)[1 - \frac{13}{H(t)}]} = s(t) \tag{77}$$

$$\frac{13 \cdot 9}{H(t) - 13} = s(t) \tag{78}$$

So the **Answer to (a)** is $s(t) = \frac{13 \cdot 9}{H(t) - 13}$.

Taking the derivative of $s(t)$, we find

$$s'(t) = \left(\frac{117}{H(t) - 13}\right)' \tag{79}$$

$$= \frac{-117}{(H(t) - 13)^2} \cdot H'(t). \tag{80}$$

Knowing that $H(t) = 100 - 4.9t^2$, we have that

$$s'(t) = \frac{-117}{(H(t) - 13)^2} \cdot H'(t) \tag{81}$$

$$= \frac{-117}{(100 - 4.9t^2 - 13)^2} \cdot (-9.8t) \tag{82}$$

$$= \frac{117 \cdot 9.8t}{(87 - 4.9t^2)^2}. \tag{83}$$

Plugging in $t = 2$, we find

$$s'(2) = \frac{117 \cdot 9.8t}{(87 - 4.9t^2)^2} \tag{84}$$

$$= \frac{117 \cdot 9.8 \cdot 2}{(87 - 4.9 \cdot 4)^2} \tag{85}$$

$$= \frac{117 \cdot 9.8 \cdot 2}{(87 - 19.6)^2} \tag{86}$$

$$= \frac{117 \cdot 9.8 \cdot 2}{(87 - 19.6)^2} \tag{87}$$

$$= \frac{117 \cdot 9.8 \cdot 2}{(67.4)^2}. \tag{88}$$

(II) The dome's radius R is constant. We find that

$$V'(t) = \left(\pi h(R^2 - \frac{h^2}{3})\right)' \tag{89}$$

$$= \left(\pi h R^2 - \frac{\pi h^3}{3}\right)' \tag{90}$$

$$= \pi h'(t) R^2 - \pi h^2 h'(t) \tag{91}$$

$$= \pi h'(t)(R^2 - h(t)^2). \tag{92}$$

(a) We are told in the problem that $V'(t) = 10$ and $h(t) = 1$. Thus we find

$$10 = \pi h'(t)(R^2 - 1^2).$$

Rearranging terms, we conclude

$$h'(t) = \frac{10}{\pi(R^2 - 1)}.$$

(b) Using the same $V'(t)$, but this time plugging in $h(t) = 2$, we find

$$h'(t) = \frac{10}{\pi(R^2 - 4)}.$$

(c) We see that

$$h'(t) = \frac{10}{\pi(R^2 - h(t)^2)}.$$

In other words, as $h(t)$ is increasing, the denominator in the fraction is decreasing—hence the rate of change of height, $h'(t)$, also increases as $h(t)$ increases.

(III) The region affected is a circle of radius $r(t)$. According to the problem, because Raleigh waves travel at a speed of 3.4 kilometers per second,

$$r(t) = 3.4t$$

where t is in seconds, and $r(t)$ is in kilometers.

(a) The area affected is

$$\pi r(t)^2 = \pi(3.4 \times 10)^2 = \pi \times 3.4 \times 10^2$$

kilometers squared.

(b) The rate of change of area can be calculated as

$$\left(\pi r(t)^2\right)' = \pi \cdot 2r(t) \cdot r'(t). \tag{93}$$

Note that we know $r'(t) = 3.4$; and part (b) does not tell us how many seconds the wave has traveled, but it does ask about the rate of change of area when $r(t) = 1$. Hence we find

$$\begin{aligned} \left(\pi r(t)^2\right)' &= \pi \cdot 2r(t) \cdot r'(t). && (94) \\ &= \pi \cdot 2 \times 1 \cdot 3.4 && (95) \\ &= 6.8\pi. && (96) \end{aligned}$$

The answer to part (b) is hence 6.8π kilometers squared per second.

5 Taylor Polynomial Problems

For each of the functions $f(x)$ and each a indicated, find a degree 4 polynomial T so that

1. $f(a) = T(a)$,
2. $f'(a) = T'(a)$,
3. $f''(a) = T''(a)$,
4. $f^{(3)}(a) = T^{(3)}(a)$,
5. $f^{(4)}(a) = T^{(4)}(a)$.

Here are the functions:

(I) $f(x) = \sin(x)$ and $a = \pi/2$.

(II) $f(x) = e^x$ and $a = 0$.

(III) $f(x) = \sqrt{(x)}$ and $a = 4$.

5.1 Solutions

(I) $f(x) = \sin(x)$ and $a = \pi/2$.

We know (see the class notes from Lecture 18) that a Taylor polynomial of degree 4 will be given by the formula

$$T(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4.$$

First let's note that since $f(x) = \sin(x)$, we have

$$f'(x) = \cos(x), \qquad f''(x) = -\sin(x), \qquad f^{(3)}(x) = -\cos(x), \qquad f^{(4)}(x) = \sin(x).$$

Now let's remember from trigonometry that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$. Thus we find

$$f(\pi/2) = 1, \qquad f'(\pi/2) = 0, \qquad f''(\pi/2) = -1, \qquad f^{(3)}(\pi/2) = 0, \qquad f^{(4)}(\pi/2) = 1.$$

Thus, we find

$$T(x) = 1 + \frac{-1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{24}(x - \frac{\pi}{2})^4.$$

(II) $f(x) = e^x$ and $a = 0$.

We must compute the derivatives of f at $a = 0$. We see

$$f'(x) = e^x, \qquad f''(x) = e^x, \qquad f^{(3)}(x) = e^x, \qquad f^{(4)}(x) = e^x.$$

Hence

$$f'(0) = 1, \qquad f''(0) = 1, \qquad f^{(3)}(0) = 1, \qquad f^{(4)}(0) = 1.$$

We conclude

$$T(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

(III) $f(x) = \sqrt{(}x)$ and $a = 4$.

We again compute the higher derivatives of f. Let's write $f(x) = \sqrt{x} = x^{1/2}$. Using the power rule over and over, we find

$$\begin{aligned}
f'(x) &= (x^{1/2})' && (97)\\
&= \frac{1}{2}x^{-1/2} && (98)\\
&= \frac{1}{2}\frac{1}{\sqrt{x}}. && (99)\\
f''(x) &= (\frac{1}{2}x^{-1/2})' && (100)\\
&= \frac{-1}{4}x^{-3/2} && (101)\\
&= \frac{-1}{4}\frac{1}{\sqrt{x}^3}. && (102)\\
f^{(3)}(x) &= (\frac{-1}{4}x^{-3/2})' && (103)\\
&= \frac{3}{8}x^{-5/2} && (104)\\
&= \frac{3}{8}\frac{1}{\sqrt{x}^5}. && (105)\\
f^{(4)}(x) &= (\frac{3}{8}x^{-5/2})' && (106)\\
&= \frac{-15}{16}x^{-7/2} && (107)\\
&= \frac{-15}{16}\frac{1}{\sqrt{x}^7}. && (108)
\end{aligned}$$

Thus, we have

$$f(4) = \sqrt{4} \tag{109}$$
$$= 2. \tag{110}$$
$$f'(4) = \frac{1}{2}\frac{1}{\sqrt{4}} \tag{111}$$
$$= \frac{1}{4}. \tag{112}$$
$$f''(4) = \frac{-1}{4}\frac{1}{\sqrt{4}^3} \tag{113}$$
$$= \frac{-1}{4}\frac{1}{8} \tag{114}$$
$$= \frac{-1}{32}. \tag{115}$$
$$f^{(3)}(4) = \frac{3}{8}\frac{1}{\sqrt{4}^5} \tag{116}$$
$$= \frac{3}{8}\frac{1}{32} \tag{117}$$
$$= \frac{3}{256}. \tag{118}$$
$$f^{(4)}(x) = \frac{-15}{16}\frac{1}{\sqrt{4}^7} \tag{119}$$
$$= \frac{-15}{16}\frac{1}{128} \tag{120}$$
$$= \frac{-15}{2048}. \tag{121}$$

Thus we have

$$T(x) = 2 + \frac{1}{4}(x-4) + \frac{1}{2}\frac{-1}{32}(x-4)^2 + \frac{1}{3!}\frac{3}{256}(x-4)^3 + \frac{1}{4!}\frac{-15}{2048}(x-4)^4 \tag{122}$$
$$= 2 + \frac{1}{4}(x-4) + \frac{-1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + \frac{1}{4!}\frac{-5}{16384}(x-4)^4. \tag{123}$$