Extra Credit Quiz 1 (Deadline February 18th)

This is worth at most 5 extra credit points.

The way this works. Come to my office hours. I will give you a fifteen-minute written quiz on the material below. You must arrive by 10:40 AM to be able to take the quiz. You must come to take the quiz during office hours on, or before, February 18th. You may not re-take the quiz.

Example. If you ace the quiz, you get 5 extra credit points. If you do terribly, you get zero points, and your overall grade for the class won't change.

How to prepare. I would suggest that you show up already knowing how to solve the problems below. In reality, it is not the quiz that is difficult—the difficulty will be in your at-home preparation, especially in figuring out how to solve the problems to begin with. If you do not overcome the at-home difficulty, the quiz will not be difficult; it will be impossible.

Exceptions. If you cannot come to any of my office hours, but still want to take the quiz, e-mail me, and we will set up a time. You must complete the quiz before 11 AM on February 18th, and my availability is limited, so e-mail me well beforehand to set up a time.

The prompt. On this extra credit quiz, I will ask you to prove one of the following limit laws *using the ϵ-δ definition of limit*.

(a) The composition law.

(b) The product law.

(c) The addition law.

Example 2.2 (Scaling law)**.** As an example, let me work out the *scaling law* for you. (Be warned: Proving the laws above are far more difficult than proving the scaling law, and you will require new insights beyond what you learn in this PDF file to prove the laws above.)

I need to prove: If $f(x)$ has a limit at a, then so does the function $m \cdot f(x)$. Moreover,

$$\lim_{x\to a}(m \cdot f(x)) = m \cdot (\lim_{x\to a} f(x)).$$

Proof.

First let's begin with the hypotheses: We know $\lim_{x\to a} f(x)$ exists by hypothesis, so let us call this limit L. Then by the ϵ-δ definition of limit, we know that for every $\epsilon > 0$, there exists a $\delta > 0$ so that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

Now, onto the proof. We must show now that $\lim_{x\to a}(m \cdot f(x))$ exists, and that it is equal to mL. Just so we don't get confused, let's use the letters e and d instead of the letters ϵ and δ. To prove $\lim_{x\to a}(m \cdot f(x)) = mL$, we must prove the following: For any $e > 0$, there exist a d such that

$$|x - a| < d \implies |mf(x) - mL| < e.$$

Well, we know that $|mf(x) - mL| < e$ if and only if $|f(x) - L| < e/|m|$. So for any given $e > 0$, let $\epsilon = e/|m|$. Note $\epsilon > 0$. So we know—because $\lim_{x\to a} f(x)$ exists—that there exists some $\delta > 0$ so that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon = e/|m|.$$

So let us set $d = \delta$. Then

$$|x-a| < d \implies |f(x)-L| < \epsilon = e/|m| \implies |m||f(x)-L| < e \implies |mf(x)-mL| < e.$$

Thus

$$|x - a| < d \implies |mf(x) - mL| < e. \tag{5}$$

We have thus shown that, for any $e > 0$, there exists $d > 0$ for which (5) holds. BY definition of limit, this shows that

$$\lim_{x\to a}(mf(x)) = mL.$$

By definition of L, the righthand side is $m \cdot \lim_{x\to a} f(x)$, so we are finished.

Remark 2.3. The symbol "$\implies$" stands for the word "implies." So "$p \implies q''$ means "p implies q," or in plain English, "If p is true, then q is true."

Here are some correct examples of the use of $\implies$:

1. R is a square $\implies$ R is a rectangle. (That is, any square is a rectangle.)

2. x is an even number $\implies$ x is a multiple of 2. (That is, any even number is a multiple of 2.)

3. You wake up at 9 AM next Monday $\implies$ You are late to class next Monday. (That is, if you wake up at 9 AM next Monday, then you will be late to class next Monday.)