3 Writing Assignment Due Thursday, February 13

This "writing" assignment is actually like a "hard homework problem."

Prompt. First, write a proof showing that

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0.$$

Then, write a proof showing that

$$\frac{d}{dx} \sin(x) = \cos(x).$$

Each proof should look like a string of equalities. Moreover, you should indicate *why* each equality you write is valid. If you are just using facts/techniques from precalculus, you may write "by precalculus facts" or "algebra." If you are using a limit law, you must indicate which. If you are using a definition of something (like the definition of a derivative), you must state "by definition of —."

Hints. You may use (and will have to use, in both proofs) the fact that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

For the first proof, a big hint: Try multiplying the top and bottom of your fraction by $\cos(h) + 1$.

The following are two examples of what your proof might look like, for a *different* problem.

Example. *Without using the power rule,* write a proof showing that

$$\frac{d}{dx}(x^2 + x) = 2x + 1.$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(x^2+x) &= \lim_{h\to 0} \frac{(x+h)^2+(x+h)-(x^2+x)}{h} & (6)\\
&= \lim_{h\to 0} \frac{x^2+2hx+h^2+x+h-x^2-x}{h} & (7)\\
&= \lim_{h\to 0} \frac{2hx+h+h^2}{h} & (8)\\
&= \lim_{h\to 0} 2x+1+h & (9)\\
&= \lim_{h\to 0}(2x+1)+\lim_{h\to 0} h & (10)\\
&= 2x+1+\lim_{h\to 0} h & (11)\\
&= 2x+1+0. & (12)\\
&= 2x+1. & (13)
\end{aligned}$$

The first equality is the definition of derivative. The next lines, (7) and (8) are just algebra. The equality (9) is from the puncture law. We arrive at equality (10) because limits add (additive law for limits). Then (11) follows because the limit of a constant function (with respect to h) is just that constant, while (12) uses that the function $f(h) = h$ is continuous. □

Here is another way to write this proof:

Example 3.1. *Without using the power rule,* write a proof showing that

$$\frac{d}{dx}(x^2+x) = 2x+1.$$

Proof.

$$
\begin{aligned}
\frac{d}{dx}(x^2 + x) &= \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} && \text{definition of derivative} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + x + h - x^2 - x}{h} && \text{algebra} \\
&= \lim_{h \to 0} \frac{2hx + h + h^2}{h} && \text{algebra} \\
&= \lim_{h \to 0} 2x + 1 + h && \text{puncture law} \\
&= \lim_{h \to 0} (2x + 1) + \lim_{h \to 0} h && \text{additive law} \\
&= 2x + 1 + \lim_{h \to 0} h && \text{limit of a constant function} \\
&= 2x + 1 + 0. && f(h) = h \text{ is continuous} \\
&= 2x + 1. && \text{obvious}
\end{aligned}
$$

□