Lecture 2

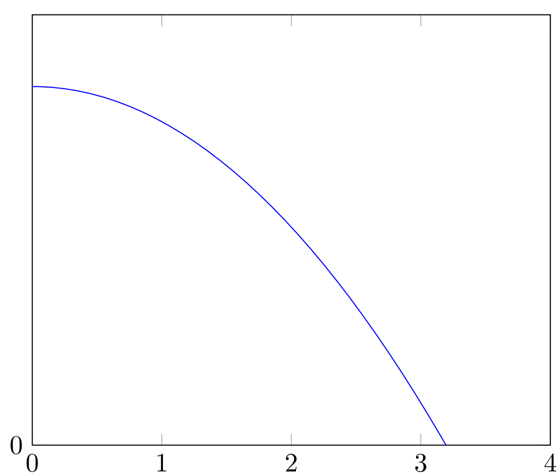
Derivatives

Last time we saw how we could try to approximate Usain Bolt's speed at $t = 5$ seconds.

Let's practice on an arbitrary function.

2.1 Speed of a falling ball

Below is a graph depicting the height $f(t)$ of a ball at time t.



The horizontal axis is in units of seconds (s), and the vertical axis is measured in meters (m).

On purpose, I have labeled nothing along the vertical axis. All you know is that at time t, the ball has height $f(t)$.

(a) Suppose your friend tells you the height of the ball after 2 seconds, and after $2 + h$ seconds. (Here, h is some number.) That is, your friend tells you

$f(2)$ and $f(2+h)$. In terms of $f(2)$ and $f(2+h)$, how far does the ball vertically travel between times 2 and $2+h$?

(c) Over that period of time, what is the average speed of the falling ball?

(d) What does this average speed have to do with the line segment between the point $(2, f(2))$ and the point $(2+h, f(2+h))$?

(e) How should you change h if you want to know the speed of the ball *at* time $t = 2$?

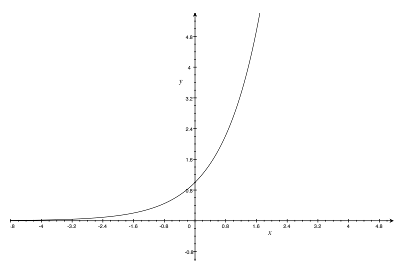
2.2 The derivative

On the previous page, you rediscovered what we have been calling the difference quotient:

$$\frac{f(x+h)-f(x)}{h}.$$

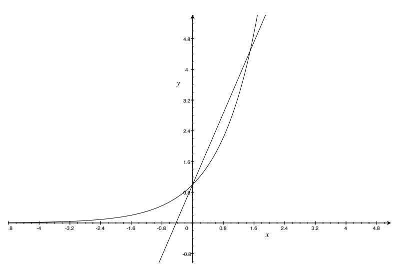
(You dealt with the special case when x equals 2.) The *geometric* interpretation of the difference quotient is as follows: When h is small, the difference quotient estimates the slope of the graph of f at the point x.

Example 2.2.1. For example, below is the graph of $f = e^x$.

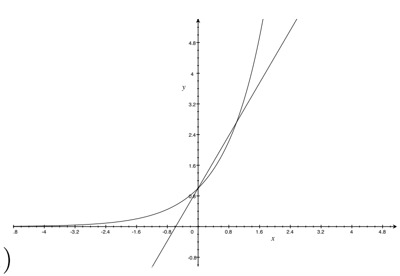


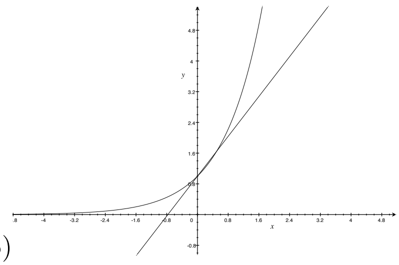
Below, we have taken smaller and smaller values of h, and drawn the line passing through the point $(0, 1)$ with slope given by the difference quotient. (Here, x is set to 0, and h is being made smaller.)

1. ($h = 1.5$)

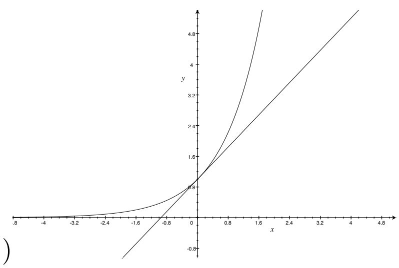


2. ($h = 1$)





3. $(h = 0.5)$



4. $(h = 0.1)$

What you are supposed to observe in this example is that, as h grows smaller, the line we draw looks more and more like the line *tangent* to the graph at the point $(0, 1)$. (End of example.)

Warning 2.2.2. The difference quotient does not make sense when $h = 0$. Why? Because then one would need to know how to divide by zero.

Warning 2.2.3 (Follow-up to previous warning)**.** If $f(x) = x^2 + 10$, then many of us have computed the following:

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} = \frac{\big((x+h)^2 + 10\big) - \big(x^2 + 10\big)}{h} &= \frac{x^2 + 2xh + h^2 + 10 - x^2 - 10}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h.\end{aligned}$$

This last line is (technically) incorrect. A more correct thing to write is:

$$\frac{2xh + h^2}{h} = 2x + h \qquad \textbf{when } h \neq 0.$$

After all, the lefthand side is only defined when h does not equal zero.

Question 2.2.4. We are led to a natural question: Does the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

tend to, or approach, some number as we take h to be smaller and smaller (that is, as h approaches zero)?

We will define what we mean by "tend to," or "approach," in a lecture or two. Ignoring that for the moment, we have the following definition of the derivative. (One of our three main calculus concepts!)

Definition 2.2.5 (The derivative)**.** If the difference quotient tends to some number $f'(x)$ as we take h to be smaller and smaller, we call $f'(x)$ the *derivative* of f at x.

2.3 Preparation for Lecture 3

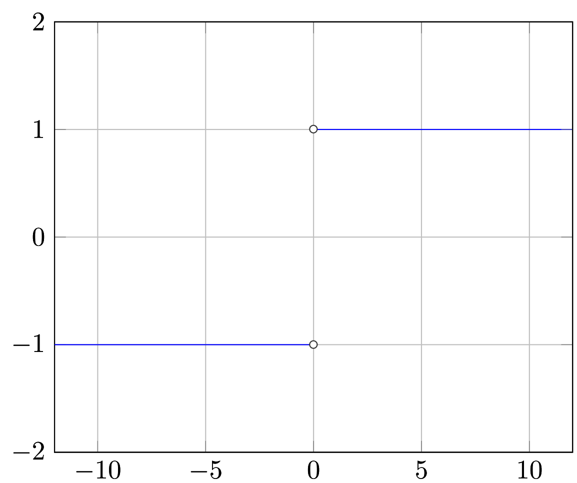
Given a function f and a point x, we will now draw the graph of the difference quotient as a function of h. (This means that the horizontal axis will be labeled by h, and the vertical axis will be labeled by the value of the difference quotient.)

Make sure to *not* draw the value at $h = 0$. (You will see why.)

Example 2.3.1. Let $f(x) = |x|$ be the absolute value function, and let $x = 0$. Then the difference quotient becomes

$$\frac{|x+h| - |x|}{h} = \frac{|0+h| - |0|}{h} = \frac{|h|}{h}.$$

This equals 1 when h is positive, and -1 when h is negative. The graph of this difference quotient is hence as follows:

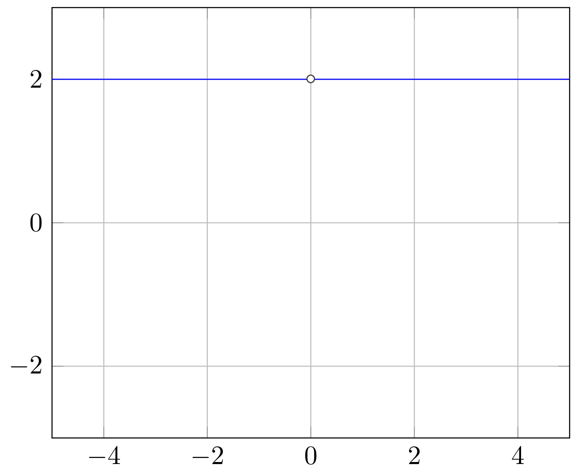


It's a good thing you didn't try to draw the value of the difference quotient at $h = 0$ here. Why?

Example 2.3.2. Let $f(x) = 2x + 1$, and choose $x = 6$. Then the difference quotient becomes

$$\frac{(2(6+h)+1) - (2(6)+1)}{h} = \frac{2h}{h}.$$

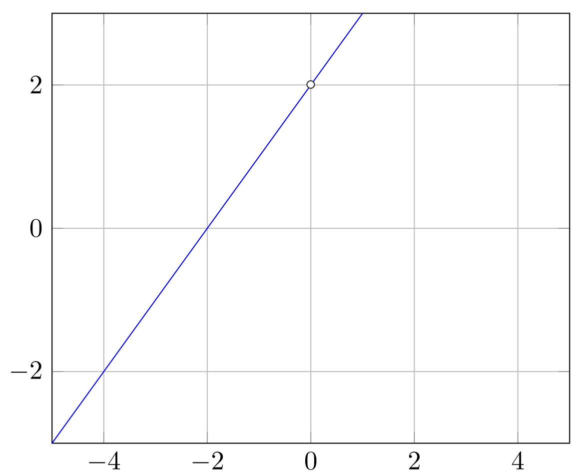
This function equals 2 when $h \neq 0$. Hence the graph of this function is as follows:



Example 2.3.3. Let $f(x) = x^2 + 2$, and choose $x = 1$. Then the difference quotient becomes

$$\frac{\big((1+h)^2+2\big) - \big((1)^2+1\big)}{h} = \frac{h^2+2h}{h}.$$

This function equals $h + 2$ when $h \neq 0$. Hence the graph of this function is as follows:



For next quiz, you should be able to draw the difference quotient (as a function of h) for the following functions f and values of x:

1. $f(x) = 3|x|$, with $x = 0$.
2. $f(x) = x - 1$, with $x = 2$.
3. $f(x) = x^2 + 1$, with $x = -1$.