Lecture 5

Continuity and more limit laws

You have entered the journey of mathematical maturity that everybody has to go through: You're being given abstract definitions, but you don't still understand what they mean in concrete situations, nor why they're useful.

5.1 Practice with the straightforward limit laws

Here are some limit laws we didn't have time to practice last time. Get in your groups and try them out.

Exercise 5.1.1. Using the limit laws, convince yourself that if $h(x) = x^2$, then

$$\lim_{x \to a} h(x) = h(a).$$

(Hint: Use the functions $f(x) = x$ and $g(x) = x$, along with the product law.)

Exercise 5.1.2. Using the limit laws, show that **limits subtract**.

More precisely, if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then so does $\lim_{x \to a} (f(x) - g(x))$. Moreover,

$$\lim_{x \to a} (f(x) - g(x)) = \left(\lim_{x \to a} f(x)\right) - \left(\lim_{x \to a} g(x)\right)$$

(Hint: Use the fact that limits scale, taking your scaling constant to be $m = -1$, and use the fact that limits add.)

Exercise 5.1.3. Use the limit laws to compute

$$\lim_{x\to 1}\left(\frac{x^2+3}{x}\right).$$

What goes wrong when you try to compute the limit as $x \to 0$?

5.2 Composition law

There is another powerful way to make new functions out of old: Composition. Limits respect composition, too, so long as the outermost function is continuous at the limit of the innermost function:

Composition law. Let $g(x)$ and $f(x)$ be functions, and suppose you know that $f(x)$ is continuous at $\lim_{x\to a} g(x)$. Then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Informally, this means you can "move the limit inside" of f so long as f is continuous where it counts.

Exercise 5.2.1. Using the composition law, and your knowledge that $f(x) = x^2$ is continuous at every point[1], compute

$$\lim_{x\to 3} f(g(x))$$

if g is a function for which $\lim_{x\to 3} g(x) = \pi$.

Warning 5.2.2. To use the composition law, the "outermost" function needs to be continuous where it counts. (Re-read the composition law if this wasn't clear when you first read it!)

5.3 One-sided limits

Sometimes, a function approaches a value from the right; sometimes, the function approaches a value from the left. These values might be different!

[1] You proved this in Exercise 5.1.1!

Definition 5.3.1 (One-sided limits)**.** If $f(x)$ wants to converge to a value as x approaches a from the right, we call this value the *righthand limit* of $f(x)$ at a, and we denote this value by

$$\lim_{x\to a^+} f(x).$$

(Note the plus sign on the a.)

If $f(x)$ wants to converge to a value as x approaches a from the left, we call this value the *lefthand limit* of $f(x)$ at a, and we denote this value by

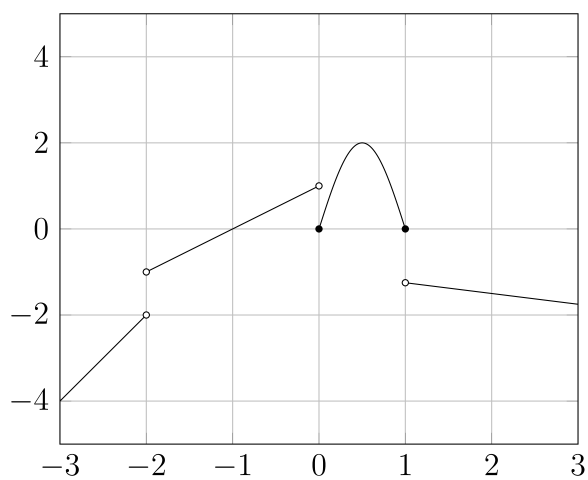
$$\lim_{x\to a^-} f(x).$$

(Note the *minus* sign on the a.)

A lefthand limit or a righthand limit is called a *one-sided limit.*

Warning 5.3.2. Just like limits, a one-sided limit may not exist!

Exercise 5.3.3. Below is the graph of a function $f(x)$.



Based on the graph, give your best guest for the following one-sided limits.

(a) $\lim_{x\to -2^-} f(x)$.

(b) $\lim_{x\to -2^+} f(x)$.

(c) $\lim_{x\to 1^+} f(x)$.

(d) $\lim_{x\to 1^-} f(x)$.

Exercise 5.3.4. Consider the function

$$f(x) = \begin{cases} 0 & x > 0 \text{ and } x \text{ is irrational} \\ 1 & x > 0 \text{ and } x \text{ is rational} \\ 13 & x < 0. \end{cases}$$

Tell me whether $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ exist, and if they exist, what their values are.

5.4 Using one-sided limits

Here is our first theorem. A *theorem* is a true statement that requires an involved proof, and the true statement is so useful that we should[2] know it for future use.

Theorem 5.4.1. The following statements are equivalent:

1. $f(x)$ has a limit at a.
2. Both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist, and the one-sided limits agree.

Moreover, in this situation, we can conclude that

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Remark 5.4.2. The term "equivalent" has a precise meaning here. It means that "if the first statement is true, then the second statement true," *and* that "if the second statement is true, then the first statement is true."

In other words, if f has a limit at a, then it has both one-sided limits there, and they agree. Conversely, if f has both one-sided limits at a and they agree, then f has a limit at a.

Example 5.4.3. Somebody tells you the following information:

$$\lim_{x \to 1^+} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 10.$$

Then you know that $\lim_{x \to 1} f(x)$ does not exist, because the two one-sided limits do not agree.

[2] That means you'll be tested on it!

Example 5.4.4. Somebody tells you the following information:

$$\lim_{x \to 2^+} f(x) = 10 \qquad \text{and} \qquad \lim_{x \to 2^-} f(x) = 10.$$

Then you know that $f(x)$ *does* have a limit at 2, because the two one-sided limits agree (that is, they have the same value). Moreover, you can conclude that

$$\lim_{x \to 2} f(x) = 10.$$

5.5 Summary of straightforward limit laws

Limits of constants. If $f(x)$ is a constant function[3] with value C, then

$$\lim_{x \to a} f(x) = C$$

regardless of a.

Limits of x. For the function $f(x) = x$, we have that

$$\lim_{x \to a} f(x) = a.$$

Warning 5.5.1. In the following limit laws, you must *already know* that all the limits on the righthand side of the equality exist before being able to conclude the existence of, and compute, the limit on the lefthand side.

Limits scale.

$$\lim_{x \to a} (m \cdot f(x)) = m \cdot \left(\lim_{x \to a} f(x) \right)$$

Limits add.

$$\lim_{x \to a} (f(x) + g(x)) = \left(\lim_{x \to a} f(x) \right) + \left(\lim_{x \to a} g(x) \right)$$

Limits multiply.

$$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x) \right) \cdot \left(\lim_{x \to a} g(x) \right)$$

Limits divide.

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

so long as $\lim_{x \to a} g(x) \neq 0$.

[3] This means $f(x) = C$ for some number C. Put another way, the graph of $f(x)$ is just a flat, horizontal line.

5.6 Preparation for Lecture 6

Definition 5.6.1. A function $f(x)$ is called *continuous* if it is continuous at every point that $f(x)$ is defined.

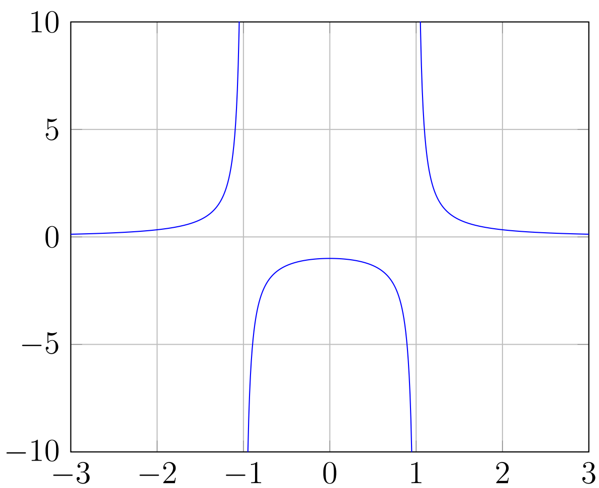
Let me give the following non-mathematical, but very helpful, intuition:

Intuition: "*A continuous function is one for which you can draw the graph of the function without ever having to lift your pencil from the paper.*"

Warning 5.6.2. This intuition fails in small ways. For example, suppose that

$$f(x) = \frac{1}{(x+1)(x-1)}.$$

Here is the graph of $f(x)$:



You can see that f is not defined at $x = 1$ and $x = -1$. So there is no way that you can draw the whole graph without lifting your pencil. But f is still a continuous function, because the value of f agrees with the limit of f at every point f is defined.

Regardless, "never have to lift your pencil" is a useful way to think about what continuity looks like. This agrees with another intuition: A continuous function has no "sudden jumps."

Example 5.6.3. As it turns out, almost every function with a "formula" that you know is continuous. Here is a list of some examples of continuous functions:

1. $f(x) = 10$ (and all other constant functions)
2. $f(x) = x$ (and all other linear functions)
3. $f(x) = 3x^3 + 4x^2 + 9$ (and all other polynomials—you can actually prove this based on the basic limit laws from last lecture)
4. $f(x) = \frac{3x^2+1}{x-3}$ (and all other functions that are quotients of polynomials—you can actually prove this based on the basic limit laws from last lecture)
5. $f(x) = |x|$ (I bet you can prove this function is continuous!)
6. $f(x) = \sin(x)$ (and all other trig functions)
7. $f(x) = \sqrt{x}$
8. $f(x) = x^p$, for any real number p. (You should be familiar with the special cases when p is a negative integer like $p = -1$ or $p = -2$, and when p is a fraction like $p = 1/3$ or $p = 2/3$.)
9. $f(x) = e^x$
10. $f(x) = \ln(x)$

The continuity of the last five examples require some proofs that we won't go over in this class.

From now on, you may use—and are expected to know—that all the functions above are continuous.

Example 5.6.4. You have now been told that $x \mapsto x^{1/n}$ is continuous. We can use the composition law to deduce the following **Root Law**: The root of the limit is the limit of the root.

That is, prove that if $\lim_{x\to a} f(x)$ exists,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

Here is the proof. Let $h(x) = x^{1/n}$. Then *because $h(x)$ is a continuous function,* so we can use the composition law to conclude that

$$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x)). \tag{5.6.0.1}$$

(Line (5.6.0.1) is where we are using the composition law.) Now let's just plug in what $h(x)$ is to simplify both sides:

$$\lim_{x\to a} h(f(x)) = \lim_{x\to a}(f(x))^{1/n} \quad , \quad h(\lim_{x\to a} f(x)) = \left(\lim_{x\to a} f(x)\right)^{1/n}. \tag{5.6.0.2}$$

Stringing (5.6.0.1) and (5.6.0.2) together, we find:

$$\lim_{x\to a}(f(x)^{1/n}) = \left(\lim_{x\to a} f(x)\right)^{1/n}. \tag{5.6.0.3}$$

And now let's just remember that raising something to the $1/n$ power is the same thing as taking the nth root. So (5.6.0.3) becomes

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

And we're done!

Warning 5.6.5. The root law only makes sense when taking nth roots makes sense. For example, if n is even, then the law only makes sense if $\lim_{x\to a} f(x)$ is not negative.

Example 5.6.6. You have now been told that $x \mapsto x^p$ is continuous. We can use the composition law to deduce the following **Power Law**: The power of the limit is the limit of the power.

That is, prove that if $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a} (f(x)^p) = \left(\lim_{x\to a} f(x)\right)^p.$$

Here is the proof. Let $h(x) = x^p$. Then *because $h(x)$ is a continuous function,* we can use the composition law to conclude that

$$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x)). \tag{5.6.0.4}$$

(Line (5.6.0.4) is where we are using the composition law.) Now let's just plug in what $h(x)$ is to simplify both sides:

$$\lim_{x\to a} h(f(x)) = \lim_{x\to a}(f(x))^p \quad , \quad h(\lim_{x\to a} f(x)) = \left(\lim_{x\to a} f(x)\right)^p. \tag{5.6.0.5}$$

Stringing (5.6.0.4) and (5.6.0.5) together, we find:

$$\lim_{x\to a}(f(x))^p = \left(\lim_{x\to a} f(x)\right)^p.$$

That's the power law we wanted to prove, so our proof is complete!

Warning 5.6.7. The power law only makes sense when taking pth powers makes sense. For example, if p is negative, then the law only makes sense if $\lim_{x \to a} f(x)$ is not zero.

For next class, you should be able to do the following:

(a) Use the composition law, and the fact that $h(x) = x^p$ is continuous, to prove the power law. **Indicate at what step you are using the composition law.**

(b) Use the composition law, and the fact that $h(x) = x^{1/n}$ is continuous, to prove the root law. **Indicate at what step you are using the composition law.**