Lecture 6

The Intermediate Value Theorem

6.1 Some warm-up exercises

(You will *not* need your preparation for this class to answer these exercises.)

Exercise 6.1.1. Consider the function $f(x) = x^2 + 10$. Does this function have a root?

(Recall that a *root* is a value of x for which $f(x)$ equals zero. So, another way to rephrase the question: is there a value of x such that $x^2 + 10$ equals zero?)

Explain.

Exercise 6.1.2. Consider the polynomial function $f(x) = x^5+7x^4-22x+19$. (This function is complicated, I know!)

Let me tell you that $f(-10)$ has the value -29,761. Also, $f(3)$ equals 763.

Based on this information, does $f(x)$ have a root?

(This question is *not* asking you to *find* a root; it's asking you whether a root *exists*.)

Explain. Can you explain in such a way where you can ignore/forget how complicated $f(x)$ looks?

6.2 The Intermediate Value Theorem (IVT)

We didn't get to go over the word "theorem" in the previous class (though it was used on the hand-out). A *theorem* is a mathematical fact that is very useful, and that somebody proved for your use. Because somebody has proven our theorems to be true[1], you may utilize theorems whenever you like in the future.

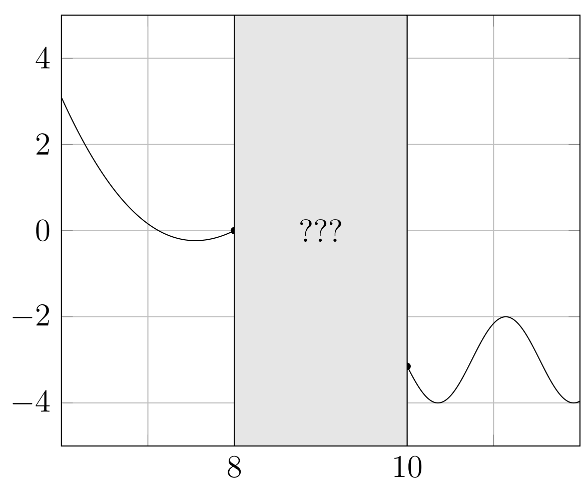
Here is a theorem.

Theorem 6.2.1 (Intermediate Value Theorem)**.** Let $f(x)$ be a continuous function, and choose two real numbers a and b with $a < b$. [2] Then for any number N between $f(a)$ and $f(b)$, [3] there is a number c between a and b so that $f(c) = N$.

Put another way, on the way from a to b, the graph of f attains (at least) every height between $f(a)$ and $f(b)$.

Remark 6.2.2. Sometimes, we abbreviate the Intermediate Value Theorem by "IVT" (especially when we are running out of time on exams or quizzes).

Example 6.2.3. Here is a graph of a function $f(x)$ that your friend began to make, then stopped part-way:



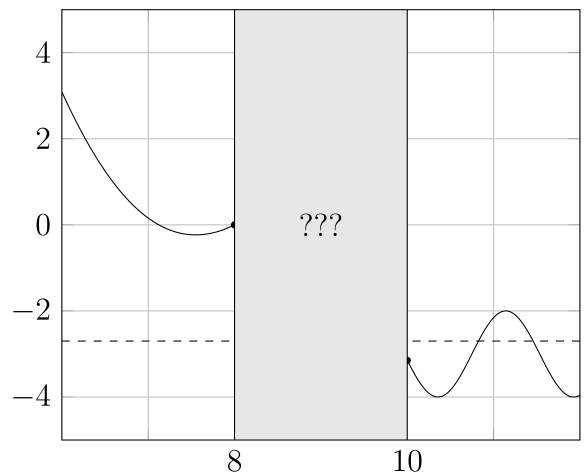
[1]The beauty is, if you want, you can prove it too! It just won't be easy with the tools you've learned so far, but you can do it.

[2]You should imagine these numbers to be on the x-axis.

[3]You should imagine N, $f(a)$, and $f(b)$ to be on the y-axis

So you have no idea what $f(x)$ looks like in the region between 8 and 10. However, you do know that $f(8) = 0$ and $f(10) = -3$. Therefore, *if* $f(x)$ *is continuous*, then the Intermediate Value Theorem tells you that $f(x)$ must hit (at least) every number between 0 and -3, at least once.[4]

For example, -2.7 is a number between 0 and -3. So, though you *do not know where*, you do know that $f(x)$ must equal -2.7 at *some value* of x between 8 and 10. [5] Here is a pictorial way to think about it:



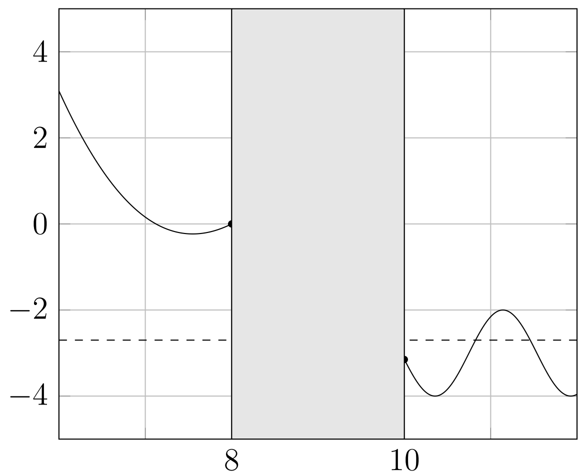
We have drawn, in dashes, the line at height -2.7. Because $f(x)$ is continuous, to get from height 0 to height -3, the graph of $f(x)$ *must* cross over this line at some point in the grey region. We don't know where $f(x)$ crosses the line, but it does so *somewhere* between $x = 8$ and $x = -10$.

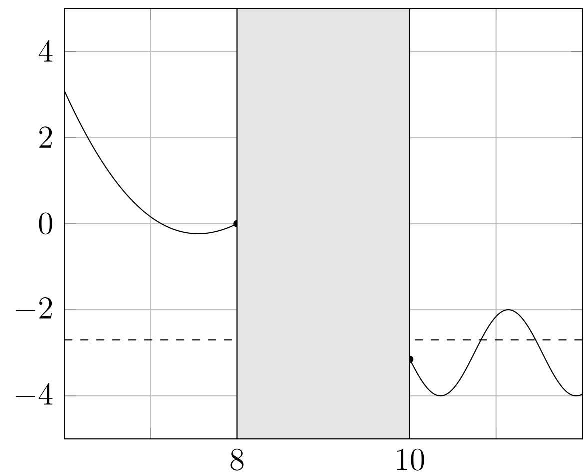
Remark 6.2.4. Note that, in Example 6.2.3, the graph of $f(x)$ crosses over the line of height -2.7 *outside* the grey region as well. That's all well and good, but the intermediate value theorem only guarantees something about the *grey region*—i.e., about the region between a and b.

Remark 6.2.5. Here are some examples of continuous functions that could fill in the grey region from Example 6.2.3:

[4] In this example, $a = 8$ and $b = 10$.

[5] In terms of the letters used in Theorem 6.2.1, $N = -2.7$. And c is the *some value* between 8 and 10.





Note that $f(x)$ may attain N at *more than one value of* c. (You can see this graphically in the lefthand example: The graph of $f(x)$ crosses the horizontal line of height $N = -2.7$ three times.)

Note that $f(x)$ *does not need to stay inbetween* $f(a)$ *and* $f(b)$. (You can see this on the righthand example.) That is, even if $a < c < b$, it need *not* be true that $f(c)$ is between $f(a)$ and $f(b)$.

Exercise 6.2.6. Do Exercise 6.1.2 again, using the IVT. Make sure you know what the values of a, b, and N are.

Do you know the value of c?

6.3 Intermediate value theorem on a closed interval

Recall that a *closed* interval is an interval of the form

$$[a, b]$$

with $a < b$. For example, $[2, 7]$ is the interval of all numbers between 2 and 7, *including* 2 and 7.

An *open* interval is an interval of the form

$$(a, b)$$

with $a < b$. For example, $(2, 7)$ is the interval of all numbers between 2 and 7, *not including* 2 and 7.

If a function $f(x)$ is defined only on a closed interval $[a, b]$, it's not obvious what we mean for f to be continuous—mainly because we can only define a one-sided limit (and not a limit) at a and b. But we take what we can get:

Definition 6.3.1. If a function $f(x)$ is defined only on a closed interval $[a, b]$, we say that f is *continuous at a* if

1. The righthand limit $\lim_{x\to a^+} f(x)$ exists, and
2. $\lim_{x\to a^+} f(x) = f(a)$.

Likewise, we say that f is *continuous at b* if

1. The lefthand limit $\lim_{x\to b^-} f(x)$ exists, and
2. $\lim_{x\to b^-} f(x) = f(b)$.

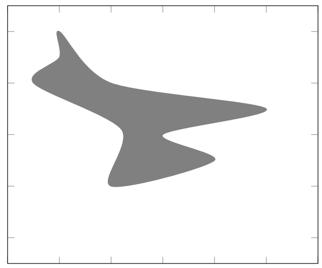
We say that f is continuous if it is continuous at every point of $[a, b]$. [6]

Theorem 6.3.2. The intermediate value theorem holds for continuous functions defined on a closed interval.

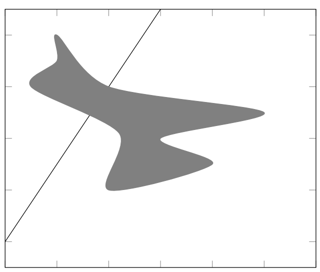
[6]Note that for any element c inside of (a, b)—that is, for any c with $a < c < b$—we know what it means for $f(x)$ to be continuous at c, because we know how to define the limit of f at c.

6.4 A fun exercise: Wonky pizza

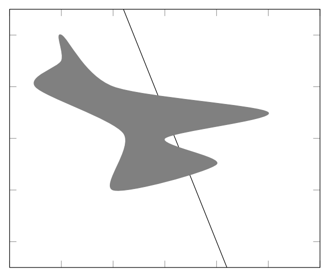
Here is a picture of a wonky-shaped pizza. (And yes, it's gray; not the most tasty-looking thing, is it?)



Your boss wants you to cut this pizza in half, using *one*, linear cut. For example,



and



are two cuts you're allowed to make. Notice that the resulting pizza can have more than just two pieces (as seen on the righthand cut). All that your boss wants is that all the pizza on one side of the cut, has the same area as all the pizza on the other side of the cut.

Exercise 6.4.1. Using the Intermediate Value Theorem, convince yourself that for *any* slope m you choose, you can make a cut of slope m such that you divide the pizza into equal halves (just as your boss requires).

Does the theorem tell you *where* to cut the pizza?

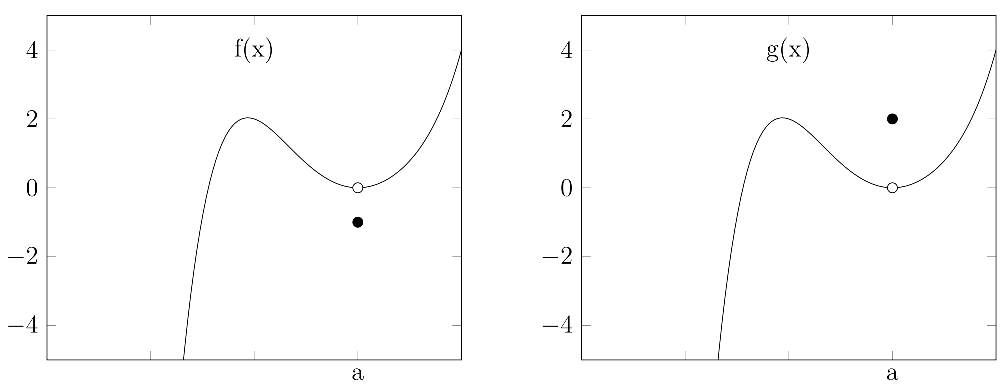
6.5 Preparation for Lecture 7

Puncture law. Let $f(x)$ and $g(x)$ be two functions. Suppose that the two function are equal away from a. Then $f(x)$ has a limit at a if $g(x)$ does, and likewise, $g(x)$ has a limit at a if $f(x)$ does. Moreover,

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Warning 6.5.1. Many calculus textbooks do *not* talk about a "puncture law." In my opinion, this is a bit ludicrous, because about half of the algebraic "tricks" we have to compute limits are dependent on it. I must admit that I made up the term "puncture law," so you may find your peers outside of your class being confused if you use this law.

Example 6.5.2 (A graphical example)**.** On the left is a graph of $f(x)$, and on the right is a graph of $g(x)$.



Note that the value of $f(x)$ and $g(x)$ are different at a (the black dots are at different heights).[7] But $f(x)$ and $g(x)$ are otherwise identical, so they have the same limit at a. This "obvious" fact is called the puncture law.

Example 6.5.3 (Algebraic example)**.** Let

$$f(x) = \frac{x^2}{x} \qquad \text{and} \qquad g(x) = x.$$

[7] Let me remind you—as I mentioned in class—that the white dot means that the function does *not* take the value of the white dot there. The black dot indicates the value of the function. Often, we write a white dot where it looks like a function wants to take a value, but does not.

Note that $f(x)$ is not defined at $x = 0$, but is equal to $g(x)$ for all other values of x. Thus, the puncture law tells us that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x). \tag{6.5.0.1}$$

Of course, you know what the righthand side is (by plugging in what $g(x)$ is):

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0. \tag{6.5.0.2}$$

So, putting (6.5.0.1) and (6.5.0.2) together, we see that

$$\lim_{x\to 0} f(x) = 0.$$

In other words (by plugging in the definition of $f(x)$) we find:

$$\lim_{x\to 0} \frac{x^2}{x} = 0.$$

Note that this is an example where the quotient law wouldn't help you, because the limit of the denominator equals zero!

Example 6.5.4 (Rational functions)**.** Let's find the limit

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x-2}.$$

Note that the function we have is *undefined* when $x = 2$ (because we can't divide by $x - 2$ when $x = 2$). But, we know the following:

$$\frac{(x+1)(x-2)}{x-2} = x + 1 \qquad \textit{so long as } x \neq 1.$$

In other words, the two functions

$$\frac{(x+1)(x-2)}{x-2} \qquad \text{and} \qquad x+1$$

are *equal* away from $x = 1$. Thus, the puncture law tells us

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2}(x+1).$$

Now, let's just compute the righthand side:

$$\begin{aligned}\lim_{x\to 2}(x+1) &= \lim_{x\to 2} x + \lim_{x\to 2} 1 \qquad (6.5.0.3)\\ &= 2+1\\ &= 3.\end{aligned}$$

(We used the addition law in line (6.5.0.3).) Putting everything together, we conclude:

$$\frac{(x+1)(x-2)}{x-2} = 3.$$

We're done, but let me streamline everything to show you what you might be able to write on a test:

$$\begin{aligned}\lim_{x\to 2}\frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law}\\ &= \lim_{x\to 2} x + \lim_{x\to 2} 1 && \text{by the addition law}\\ &= 2+1\\ &= 3.\end{aligned}$$

Another solution you might write on a test is:

$$\begin{aligned}\lim_{x\to 2}\frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law}\\ &= 2+1 && \text{because polynomial functions are continuous}\\ &= 3.\end{aligned}$$

Example 6.5.5 (Another rational function)**.** Let's do another rational function example. Let's compute [8]

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9}.$$

This looks very complicated; to use the puncture law, we'd like to find some other function that is equal to $\frac{x^2-2x-3}{x^2-9}$ away from 3. The trick I want you to learn here is that you can *cancel* $(x-3)$ in the top and bottom. This

[8] Note that the quotient law doesn't help here, because the limit of the denominator equals zero.

may seem very confusing, because $(x-3)$ doesn't appear anywhere in the function as it's presented. But you'll see that it *does* appear if you factor.

Pro tip. Why do you want to try to cancel $x-3$? It's because we should feel that a term of the form "$x-3$" is what's causing the denominator to equal zero at $x=3$. So it's natural to try and see if, indeed, a factor of $(x-3)$ can pop up in the denominator. More generally, for rational functions, *if you are computing a limit as x approaches a, it is natural to try to find $(x-a)$ as a factor of the top and bottom.*

Warning. If you don't know how to divide or factor polynomials, you should learn by Googling online and practicing—in this class, you are already *expected* to know how to divide polynomials using long division, or to factor polynomials through other tricks).

In fact, we can factor both the top and the bottom:

$$\frac{x^2-2x-3}{x^2-9} = \frac{(x-3)(x+1)}{(x-3)(x+3)}.$$

And we see that we can cancel the $(x-3)$ terms! So, when x does not equal 3, our function $\frac{x^2-2x-3}{x^2-9}$ is equal to

$$\frac{x+1}{x+3}. \tag{6.5.0.4}$$

By the puncture law, we thus conclude the following:

$$\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9} = \lim_{x\to 3} \frac{x+1}{x+3}.$$

And, as we saw in the preparation for last lecture, *any rational function is continuous where it is defined.* The rational function in (6.5.0.4) is defined at $x=3$, so—by the definition of continuity—we can compute the limit simply by plugging 3 into x:

$$\lim_{x\to 3} \frac{x+1}{x+3} = \frac{3+1}{3+3} = \frac{4}{6} = \frac{2}{3}.$$

Putting everything together, we conclude

$$\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9} = 2/3.$$

For next class's quiz, I expect you to be able to use the puncture law to compute limits of rational functions. For example, you should be able to compute the following limits:

1. $\lim_{x\to 0} \frac{x^3+3x^2}{x^2}$.
2. $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.
3. $\lim_{x\to -2} \frac{x^2-4}{x^2+x-2}$.