Lecture 7

Derivatives

Let f be a function. You have already seen the following definition, but I will state it again as a reminder:

Definition 7.0.1. The *derivative of f at x* is the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If this limit does not exist, we say that f does *not* have a derivative at x.

Definition 7.0.2. Some more terminology: If f has a derivative at x, we also say that f is *differentiable at x*. If f is differentiable everywhere f is defined, we say that *f is differentiable*.

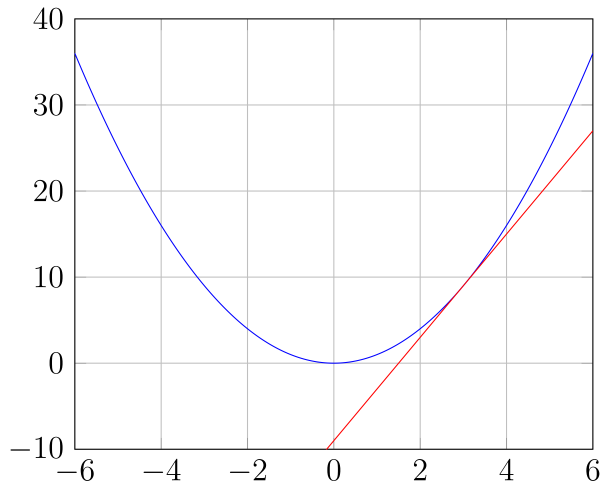
We can see why we spent so much time studying the difference quotient, and studying limits. We need to be comfortable with difference quotients to write the limit we want to compute. And we need to be comfortable with limits to be able to compute the limit we wrote!

Warning 7.0.3. The limit is computed as h goes to zero, *not* as x goes to zero.

Remark 7.0.4. Remember that the difference quotient measures the slope of the "secant" line going through $f(x)$ and $f(x+h)$. The *limit* as h goes to zero measures the slope of the "tangent" line at x.

7.1 From the basics

Exercise 7.1.1. Below is the graph of $f(x) = x^2$. I have also drawn the tangent line to $f(x)$ at $x = 3$.



Using the definition of the derivative (and the limit laws), tell me the slope of this tangent line.

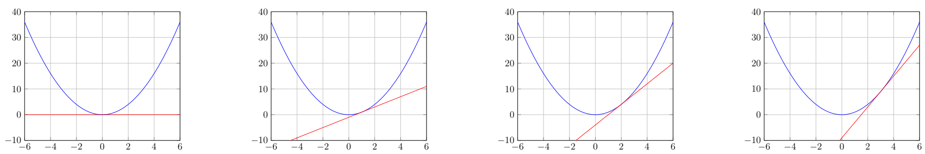
Exercise 7.1.2. Let $f(x) = 3x + 2$.

1. Find the derivative of $f(x)$ at $x = 2$ by drawing the graph of $f(x)$.
2. Find the derivative of $f(x)$ at $x = 2$ by using the definition of the derivative.

Warning 7.1.3. It is *very rare* that you'll be able to accurately compute the derivative of a function by *drawing* the graph—you can basically only eyeball a derivative when the function looks "flat" somewhere (where the derivative is zero). A computer will be able to approximate the derivative very well, but it is hard for us to do with the naked eye—at least, it's hard without some very accurate estimates. So it's a good thing we studied how to take limits *algebraically*—i.e., using formulas.

7.2 The derivative as a function (important notations!)

Let $f(x)$ be a function. Then for every x, we can ask what the slope of the tangent line at x is. Consider the example $f(x) = x^2$. I have drawn tangent lines for several values of x below:



These are plotted at $x = 0, 1, 2, 3$ respectively.

As you can see, the slope of the tangent line *changes* as the value of x changes.

Well, if for every x, we get a number called the slope, we have a new function! [1] We call this function the *derivative* of f.

Definition 7.2.1. Suppose $f(x)$ is differentiable at every x where f is defined. Then the *derivative of* f is the function f' defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The lefthand side is read "f-prime of x." The word "prime" refers to the little tick mark $'$ that you see in $f'(x)$.

Notation 7.2.2. Sometimes, we will write

$$\frac{df}{dx}$$

instead of $f'(x)$. That is, $\frac{df}{dx}$ *and* f' *are the same thing.* Because we so often write $y = f(x)$, you may also see notation such as

$$\frac{dy}{dx}$$

[1]Remember that a *function* is just a way of producing an output number from an input number. In this case, the input is x, and the output is the slope (of the tangent line at x).

and

$$y'$$

and I want you to know that *these all mean the same thing.*

Notation 7.2.3. We will also think of "taking the derivative" as a way to take one function and output another. That is, we begin with a function $f(x)$ and we produce a new function called $f'(x)$.

This operation of taking the derivative will be notated by

$$\frac{d}{dx}$$

so, for example,

$$\frac{d}{dx}f = \frac{df}{dx} = f'.$$

I know this all seems pedantic and useless, but these notations are all commonly used, so I want you to see this notation now to prevent future confusion.

Warning 7.2.4. Before, we were a little cavalier/careless about the distinction between functions and numbers. For example, if f is the name of a function, then $f(x)$ is actually the *number* that f outputs for x.

I have been careless about this as well—I'll often say "let $f(x)$ be a function."

From now on, it will be *very important* to know what is a function, and what is an output number.

7.3 Basic laws for derivatives

Just as for limits, we will first compute derivatives for the *simplest examples*, and then find ways to compute more complicated derivatives.

Constant functions have zero derivative. That is, if $f(x) = C$ for some real number C, then

$$f'(x) = 0.$$

Just to get practice with notation, this could also be written

$$\frac{d}{dx}(C) = 0.$$

Proof. Let's see that "constant functions have zero derivative" in two ways.

1. Graphically, $f(x) = C$ has a graph given by a horizontal line at height C. This line clearly has slope zero, and the tangent line at any point is (perhaps confusingly) the same horizontal line at height C! So the tangent lines also have slope zero everywhere.

2. Or, using the definition of the derivative, we can compute:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (7.1)\\ &= \lim_{h\to 0} \frac{C - C}{h} && (7.2)\\ &= \lim_{h\to 0} \frac{0}{h} && (7.3)\\ &= \lim_{h\to 0} 0 && (7.4)\\ &= 0. && (7.5) \end{aligned}$$

The first equality is the definition of the derivative. The next line follows from plugging in the definition of f. The next line is algebra. Then we use the puncture law, and finally we use that a limit of a constant function (with value zero) is just the value of that function. □

The identity function has derivative 1. That is, if $f(x) = x$, then

$$f'(x) = 1.$$

Proof. We can again see this in two ways.

1. Graphically, $f(x) = x$ has a graph given by a line of slope 1. So the tangent line at any point is (perhaps confusingly) the same line of slope 1.

2. Or, using the definition of the derivative, we can compute:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (7.6)\\ &= \lim_{h\to 0} \frac{x+h-x}{h} && (7.7)\\ &= \lim_{h\to 0} \frac{h}{h} && (7.8)\\ &= \lim_{h\to 0} 1 && (7.9)\\ &= 1. && (7.10) \end{aligned}$$

The first equality is the definition of the derivative. The next line follows from plugging in the definition of f. The next line is algebra. Then we use the puncture law, and finally we use that a limit of a constant function (with value 1) is just the value of that function. □

Derivatives scale. That is, if f is differentiable at x, and m is a number, then

$$\left(\frac{d}{dx}(mf)\right)(x) = m\frac{df}{dx}(x).$$

Another way to write this would be

$$(mf)'(x) = mf'(x).$$

Example 7.3.1. Let's find the derivative of $h(x) = 7x$. Set $f(x) = x$ and $m = 7$. Then

$$\begin{aligned} \left(\frac{d}{dx}h\right)(x) &= \left(\frac{d}{dx}(mf)\right)(x) && (7.11)\\ &= m\frac{df}{dx}(x) && (7.12)\\ &= 7 \cdot 1 && (7.13)\\ &= 7. && (7.14) \end{aligned}$$

The first line uses the fact that $h = f + g$, and the next line uses the addition law for derivatives. The last line uses the fact that we know—when $f(x) = x$ and $g(x)$ is constant—that $f'(x) = 1$ and $g'(x) = 0$.

Derivatives add. That is, if f and g are two functions, and if they both are differentiable at x, then

$$\left(\frac{d}{dx}(f+g)\right)(x) = \frac{df}{dx}(x) + \frac{dg}{dx}(x).$$

Another way to write this would be

$$(f+g)'(x) = f'(x) + g'(x).$$

Example 7.3.2. Let's find the derivative of $h(x) = 7x + 3$. Set $f(x) = 7x$ and $g(x) = 3$. Then

$$\left(\frac{d}{dx}h\right)(x) = \left(\frac{d}{dx}(f+g)\right)(x) \tag{7.15}$$

$$= \frac{df}{dx}(x) + \frac{dg}{dx}(x) \tag{7.16}$$

$$= 7 + 0. \tag{7.17}$$

The first line uses the fact that $h = f + g$, and the next line uses the addition law for derivatives. The last line uses the fact that we now know—when $f(x) = 7x$ and $g(x)$ is constant—that $f'(x) = 7$ and $g'(x) = 0$.

7.4 The power law

At this point, derivatives follow *very* different rules than limits:

The power rule *(for powers of x).* Let n be any integer, and let $f(x) = x^n$. Then f is differentiable at any x, and

$$\left(\frac{d}{dx}(f)\right)(x) = nx^{n-1}.$$

Another way to write this would be

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Warning 7.4.1. The power rule for derivatives is very different from the power rule for limits. One thing to notice is that we are only taking powers of x; we are not taking powers of more complicated functions. That is, we have not yet dealt with derivatives of things like $(g(x))^n$ for an arbitrary function g. (This will be in our next classes, when we law the "chain rule" for derivatives.)

Also, the derivative does *not* simply "move inside" the parentheses in this power rule. Put another way, $\frac{d}{dx}(x^n) \neq (\frac{d}{dx}(x))^n$.

Example 7.4.2. Let's find the derivative of $f(x) = x^3 + x^2 + 3$. We have:

$$\frac{d}{dx}(x^3 + x^2 + 3) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) + \frac{d}{dx}(3) \tag{7.18}$$
$$= 3x^2 + 2x + 0 \tag{7.19}$$
$$= 3x^2 + 2x. \tag{7.20}$$

The first line uses the addition law for derivatives, and the next line uses the power law for derivatives.

You can now find the derivative of any polynomial! You will be expected to be able to do so using the power rule and the addition rule. You will get practice in lab.

Exercise 7.4.3. Compute the derivatives of the following functions:

1. $3x - 5$.
2. $3x^2 - 8x + 5$.
3. $3x^4 - 8x^2 + 7x + 2$.

7.5 The Leibniz rule

This is my favorite.

Derivatives do *not* multiply. But we *can* find the derivative of a product. The formula for doing this is called:

The Leibniz rule (also known as the product rule). If f and g are differentiable at x, then so is the product fg, and the derivative is computed as follows:

$$\left(\frac{d}{dx}(f \cdot g)\right)(x) = f'(x)g(x) + f(x)g'(x).$$

Exercise 7.5.1. Your friend comes to you—quite proud—and thinks they can prove the Leibniz rule. They say "Let $h(x) = f(x) \cdot g(x)$," and they write down the following sequence of equations:

$$\left(\frac{d}{dx}h\right)(x) \tag{7.21}$$

$$= \left(\frac{d}{dx}(f \cdot g)\right)(x) \tag{7.22}$$

$$= \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h} \tag{7.23}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \tag{7.24}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) + f(x)g(x)}{h} \tag{7.25}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h)f(x)g(x)}{h} \tag{7.26}$$

$$= \lim_{h \to 0} \left(g(x+h)\frac{f(x+h) - f(x)}{h}\right) + \lim_{h \to 0} f(x)\frac{g(x+h)g(x)}{h} \tag{7.27}$$

$$= \lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h)g(x)}{h} \tag{7.28}$$

$$= g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x) \cdot \lim_{h \to 0} \frac{g(x+h)g(x)}{h} \tag{7.29}$$

$$= g(x) \cdot f'(x) + f(x) \cdot g'(x) \tag{7.30}$$

$$= f'(x)g(x) + f(x)g'(x). \tag{7.31}$$

Is this a valid proof? Is every step correct? Should your friend say more? Discuss.

7.6 Preparation for Lecture 8

We discussed in class that

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

is an example of a limit where the quotient rule does not help us, and we'll have to do more work. For the last week, the puncture rule has helped us. Here, it will not. We deal with a new limit law to help us:

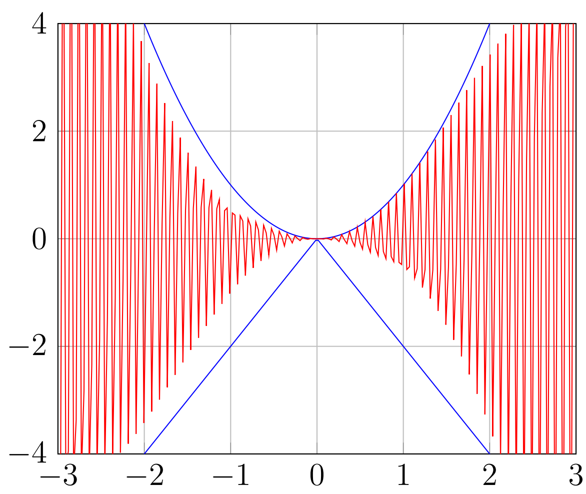
Theorem 7.6.1 (The squeeze theorem)**.** Let f, g, h be three functions, and suppose we know the following three facts:

1. For every x,
$$f(x) \leq g(x) \leq h(x),$$
2. The limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} h(x)$ exist, and
3. $$\lim_{x \to a} f(x) = \lim_{x \to a} h(x).$$

Then we can conclude that $\lim_{x \to a} g(x)$ exists, and is equal to the limits of f and h:

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x).$$

Example 7.6.2. Here is a pictorial example:



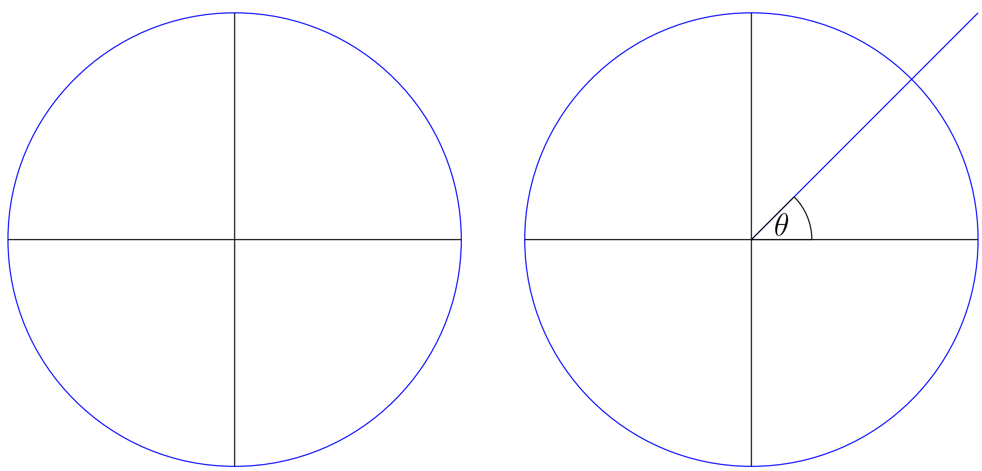
The parabola-looking graph at the top is $h(x)$; the very erratic, highly oscillatory graph in the middle is $g(x)$; and the absolute-value-like graph at the bottom is $f(x)$. In this picture, we see that

$$f(x) \leq g(x) \leq h(x)$$

for every x, simply by comparing the heights at each x. Note that f and h have limits at the origin—they both approach height 0 as x approaches zero. Hence, g is "squeezed" and forced to have a limit as x approaches zero (and in fact, the value of the limit of g there must *equal* the limits of f and h).

Our goal for next lecture will be to use the squeeze theorem to compute $\lim_{x\to 0} \frac{\sin(x)}{x}$. For this, we will set $g(x) = \sin(x)/x$. We need to find candidates for $f(x)$ and $h(x)$.

We will find these candidates shortly; it turns out we need some very clever insights to do so. Here is one of the clever insights. Consider a circle of radius 1, drawn on the left:



Now, imagine somebody gives me an angle θ (drawn on the right).

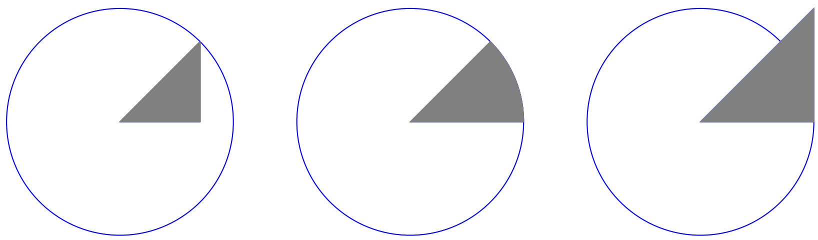
I can draw *three shapes* based on this angle:

1. A small right triangle contained inside the circle, making angle θ at the origin.

2. A sector of the circle—that is, a slice of the circular pie—making an angle θ at the origin.

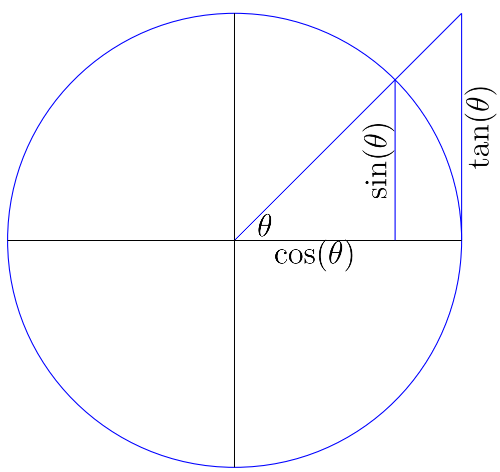
3. A large right triangle whose height is tangent to the circle, making angle θ at the origin.

These are drawn below:



I want to note the following: The smaller triangle (inside the circle) has height $\sin(\theta)$ and width $\cos(\theta)$. The larger triangle has height $\tan(\theta)$ and width 1. **You should be able to justify the claims in this paragraph.** (This is dependent on your knowledge from precalculus, or earlier in life; as you can see, you may need to brush up on your trigonometry.)

A summary is as follows, where I've drawn the circle—along with the three shapes above—all in one picture, with important lengths labeled:



(7.32)

(Remember that the circle has radius 1.)

From the picture, it is clear that

$$\text{Area of small triangle} \leq \text{Area of sector of angle } \theta \leq \text{Area of large triangle.}$$

(In fact, from the picture, we can take "$<$" instead of "$\leq$," but we won't need to use this fact.)

So let's remember that

- The area of a triangle is 1/2 times width times height, and
- The area of a sector of angle θ is given by $\pi r^2 \frac{\theta}{2\pi}$. In our example, the radius of the circle is 1, so $r = 1$.

So the above inequalities of areas becomes:

$$\frac{\sin(\theta)\cos(\theta)}{2} \leq \frac{\theta}{2} \leq \frac{\tan(\theta)}{2}. \tag{7.33}$$

For next lecture's quiz, I expect you to be able to do the following:

1. State the squeeze theorem. (That is, you should memorize Theorem 7.6.1 and be able to state it again on the quiz.)
2. Be able to produce the inequality in (7.33) by drawing the appropriate picture (7.32), and computing areas correctly. Once you draw the appropriate picture, you should be able to explain in words the three areas you are computing.