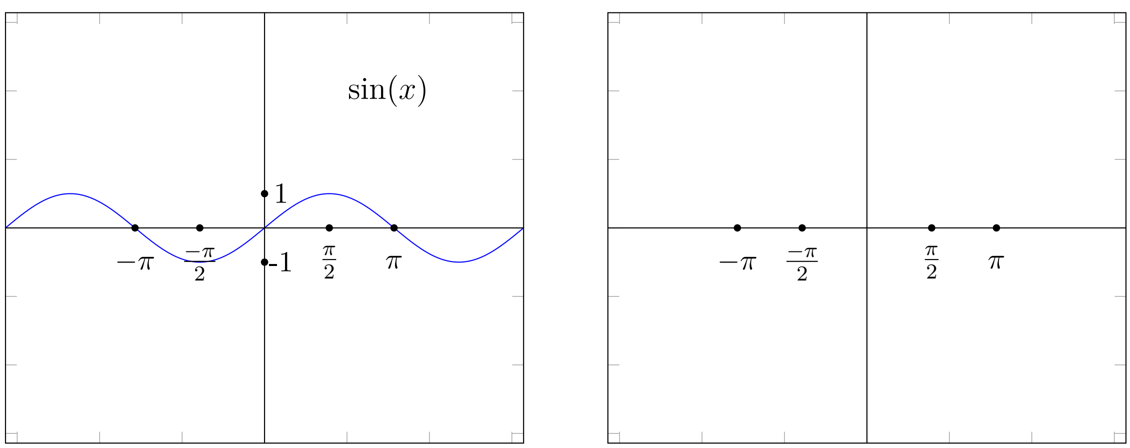
Lecture 8

Derivatives of sine and cosine

Last time we learned how to take derivatives of polynomials.

To day we'll learn how to take derivatives of $\sin(x)$ and $\cos(x)$.

Exercise 8.0.1. Below on the left is the graph of $\sin(x)$.



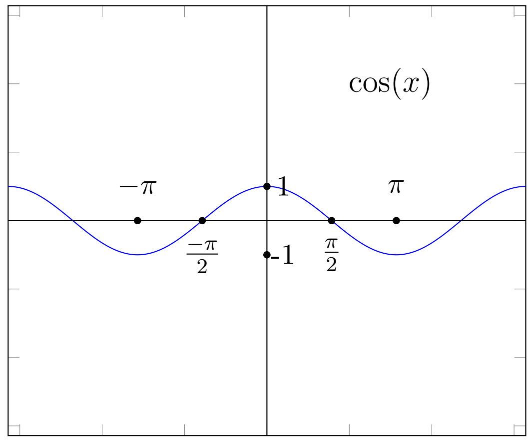
Based on the lefthand graph, **draw the graph of** $\frac{d}{dx}\sin(x)$ on the right.

No need for extreme precision, but try to do the best you can at *guessing* what the graphs of these derivatives look like.

Some tips:

- Begin by understanding where the derivative equals zero.
- Then, understand where the derivatives are positive, or negative.

Exercise 8.0.2. Below is the graph of $\cos(x)$. Does it have any relation to the picture you drew?



We have seen evidence of the derivative of sine being cosine. This turns out to be true! In fact, let me also tell you what the derivative of cosine is, too:

Theorem 8.0.3 (Derivatives of sine and cosine)**.**

$$\frac{d}{dx}(\sin)(x) = \cos(x), \qquad \frac{d}{dx}(\cos)(x) = -\sin(x).$$

Written another way,

$$(\sin)'(x) = \cos(x), \qquad (\cos)'(x) = \sin(x),$$

or

$$\sin' = \cos, \qquad \cos' = -\sin,$$

or

$$\frac{d}{dx}\sin = \cos, \qquad \frac{d}{dx}\cos = -\sin.$$

In words, *the derivative of sine is cosine. The derivative of cosine is* **negative** *sine.*

Example 8.0.4. Let's find the slope of line tangent to the graph of $\sin(x)$ at $x = \pi$. We know from above that

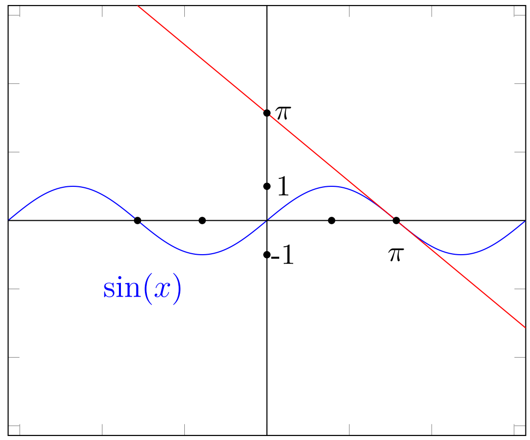
$$(\frac{d}{dx}\sin)(x) = \cos(x),$$

so

$$(\frac{d}{dx}\sin)(\pi) = \cos(\pi).$$

And now we must remember from trigonometry that $\cos(\pi) = -1$.

Here is a picture to confirm that, indeed, the tangent line at $x = \pi$ looks like it has slope -1:



(The tangent line is drawn in red.)

Exercise 8.0.5. Find the derivative of $\cos(x)$ at the following points:

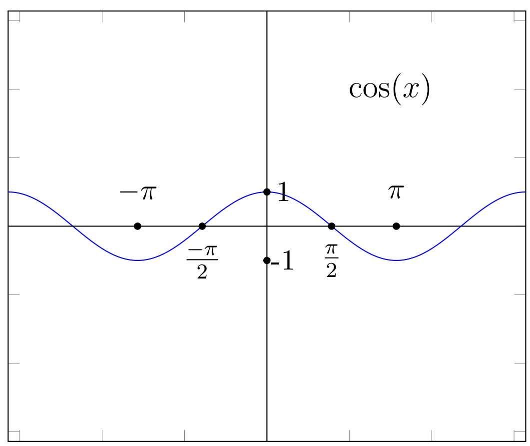
(a) $x = 0$

(b) $x = \pi/2$

(c) $x = \pi$

(d) $x = -\pi$

Do your answers make sense when you look at the graph of $\cos(x)$?



8.1 The start of proving that $\frac{d}{dx}(\sin x) = \cos x$.

I want to *prove* to you that the derivative of sine is cosine.

Before that, I need to prove two *lemmas*. A lemma is a statement that's a bit tricky to prove, but that you need to prove in order to prove a theorem.

Here is our old friend:

Lemma 8.1.1.

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

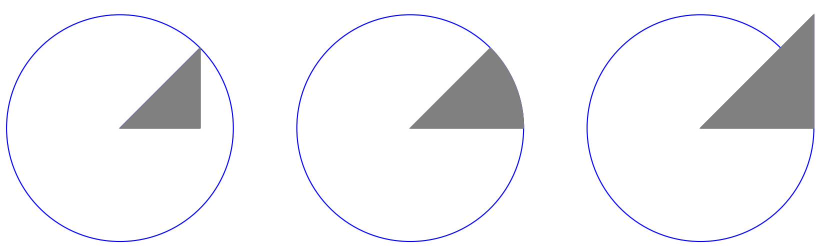
This was one example of a limit that we couldn't compute using the quotient rule, because the limit of the denominator goes to zero as $h \to 0$.

We will prove this using the squeeze theorem, which you learned about for today:

Proof of Lemma 8.1.1. For today, you were prepared with the knowledge that

$$\frac{\sin x \cos x}{2} \leq \frac{x}{2} \leq \frac{\sin x}{2\cos x}. \tag{8.1}$$

This inequality is true because these expressions are the areas of the shaded regions below, formed by the angle x:



Note two things: The angle x is *positive* in the pictures above, and also *less than* $\pi/2$—that is, less than 90 degrees. Thus, $\sin x$ is *positive*. So we can divide the inequality in (8.5) by $\sin x$ without changing the directions of the inequalities:

$$\frac{\cos x}{2} \leq \frac{x}{2\sin x} \leq \frac{1}{2\cos x}. \tag{8.2}$$

Now, note that if I have two numbers a, b satisfying $a \leq b$, then I know that $(1/a) \geq (1/b)$. So, I can "flip the fractions" and flip the inequalities. That is, (8.2) implies the following:

$$\frac{2}{\cos x} \geq \frac{2\sin x}{x} \geq 2\cos x. \tag{8.3}$$

Dividing everything by 2, I obtain

$$\frac{1}{\cos x} \geq \frac{\sin x}{x} \geq \cos x. \tag{8.4}$$

And now I use the *squeeze theorem*.[1] The outer terms of this inequality have limits as $x \to 0$, and in particular, as x approaches zero from the right:

$$\lim_{x \to 0^+} \frac{1}{\cos x} = \frac{1}{\lim_{x \to 0^+} \cos x} = \frac{1}{\cos(0)} = \frac{1}{1} = 1$$

and[2]

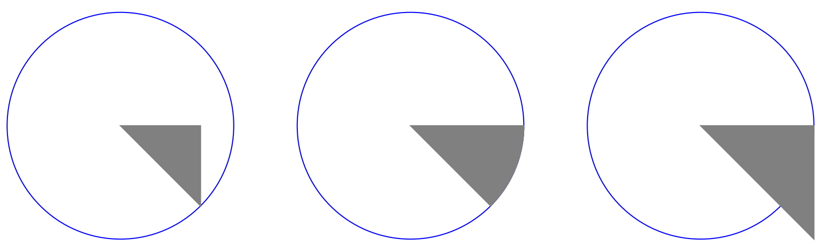
$$\lim_{x \to 0^+} \cos x = \cos(0) = 1.$$

These two limits agree (they are both 1)! So, by the squeeze theorem, we conclude

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

Now, we have to compute the *lefthand* limit of $\sin(x)/x$ and see that it equals 1; then we will have proven the lemma.

Well, when x is negative (which x must be, if it is approaching 0 from the left), the areas of the shaded regions change.



The areas become

$$\frac{-\sin x \cos x}{2} \leq \frac{-x}{2} \leq \frac{-\sin x}{2 \cos x}. \tag{8.5}$$

(One way to see this is that areas must always be *positive*, so if x is negative, we must flip the sign of $\sin x$ and x to obtain positive numbers in the inequality.)

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[1] Technically, I am using a version of the squeeze theorem for one-sided limits; this is different from the squeeze theorem in your notes. But the spirit is the same.

[2] Note that we have used our knowledge that cos is continuous to see that $\lim_{x \to 0} \cos(x) = \cos(\lim_{x \to 0} x) = \cos(0)$.

Now, dividing by $-\sin x$ (which is a *positive* number when the angle is negative!) we obtain (8.2) again, and the rest of the algebraic work we did before carries through to show that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1.$$

Thus, we see that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1. = \lim_{x\to 0^+} \frac{\sin x}{x}.$$

Because the righthand and lefthand limits agree, the limit as $x \to 0$ exists, and we conclude

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

□

You are expected to know this particular limit from now on, and you may use it at will. That is, you are expected to know

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We haven't yet proven that the derivative of $\sin(x)$ is $\cos(x)$. To complete this proof will be your writing assignment for this week!

8.2 Preparation for next time: Practice with the Leibniz rule

We have briefly seen the Leibniz rule:

The Leibniz rule (also known as the product rule). If f and g are differentiable at x, then so is the product fg, and the derivative is computed as follows:

$$\left(\frac{d}{dx}(f \cdot g)\right)(x) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)).$$

Written another way,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Example 8.2.1. Let's compute the derivative of $x^2 \sin(x)$. We have

$$\frac{d}{dx}(x^2 \sin(x)) = \left(\frac{d}{dx}(x^2)\right) \cdot \sin(x) + x^2 \cdot \left(\frac{d}{dx}(\sin(x)\right) \tag{8.6}$$

$$= 2x \cdot \sin(x) + x^2 \cos(x). \tag{8.7}$$

The first line is the Leibniz rule, and the next equality follows from our knowledge of the derivative of x^2 (the power law from last class) and of the derivative of sin (from this class).

Example 8.2.2. Let's compute the derivative of $x^3 \cos(x)$. We have

$$\frac{d}{dx}(x^3 \cos(x)) = \left(\frac{d}{dx}(x^3)\right) \cdot \cos(x) + x^3 \cdot \left(\frac{d}{dx}(\cos(x)\right) \tag{8.8}$$

$$= 3x^2 \cdot \cos(x) - x^3 \sin(x). \tag{8.9}$$

The first line is the Leibniz rule, and the next equality follows from our knowledge of the derivative of x^3 (the power law from last class) and of the derivative of cos (from this class).

Example 8.2.3. Let's compute the derivative of $(3x^2+x)(x-3)$. This won't involve any sin or cos.

There are two ways to do this. One way to do this is by using the Leibniz rule:

$$\frac{d}{dx}\Big((3x^2+x)(x-3)\Big) = \frac{d}{dx}(3x^2+x)\cdot(x-3) + (3x^2+x)\cdot\frac{d}{dx}(x-3) \tag{8.10}$$
$$= (3\cdot 2x+1)\cdot(x-3) + (3x^2+x)\cdot 1 \tag{8.11}$$
$$= (6x+1)\cdot(x-3) + (3x^2+x)1 \tag{8.12}$$
$$= 6x^2 - 17x - 3 + 3x^2 + x \tag{8.13}$$
$$= 9x^2 - 16x - 3. \tag{8.14}$$

Another way is to first multiply the factors together, and then use the addition and power rules:

$$\frac{d}{dx}\Big((3x^2+x)(x-3)\Big) = \frac{d}{dx}\Big(3x^3 - 9x^2 + x^2 - 3x\Big) \tag{8.15}$$
$$= \frac{d}{dx}\Big(3x^3 - 8x^2 - 3x\Big) \tag{8.16}$$
$$= 3\cdot 3x^2 - 2\cdot 8x - 3 \tag{8.17}$$
$$= 9x^2 - 16x - 3. \tag{8.18}$$

For next lecture, I expect you to be able to do any of the following exercises:

Exercise 8.2.4. Compute the derivatives of the following functions:

1. $-\cos(x)$
2. $x - \cos(x)$
3. $-x\cos(x)$
4. $-x^2\cos(x)$.
5. $\sin(x)^2$.
6. $\sin(x)\cos(x)$.
7. $x^3 + 3x - 2$.
8. $(x-3)(x-2)$.

9. $(x^2 - 1)(3x - 1)$.

Exercise 8.2.5. What is the slope of the tangent line to the graph of $f(x) = x\cos(x)$ at $x = \pi/4$?