Lecture 9

The chain rule

Here are the summaries of the rules we know for limits and derivatives so far:

Rule	For limits	For derivatives
Constants	$\lim_{x\to a} C = C$	$\frac{d}{dx}(C) = 0.$
Scaling	$\lim_{x\to a} mf(x) = m\lim_{x\to a} f(x)$	$\frac{d}{dx}(mf) = m\frac{d}{dx}f$
Sums	$\lim_{x\to a} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} +g(x)$	$\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$
Powers	$\lim_{x\to a} x^n = a^n$	$\frac{d}{dx}(x^n) = nx^{n-1}.$
Products	$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x).$	$\frac{d}{dx}(fg) = f'g + fg'.$
Quotients	$\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$	???
Composition	$\lim_{x\to a}(f(g(x)) = f(\lim_{x\to a} g(x)).$	???

Today, we are going to practice taking derivatives of *compositions*. You will utilize quotients in your preparation for next class.

The formula for computing derivatives of compositions is called the *chain rule*.

Theorem 9.0.1 (Chain rule)**.** Suppose that g is differentiable at x, and that f is differentiable at $g(x)$. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Put another way,

$$(\frac{d}{dx} f \circ g)(x) = (\frac{d}{dx} f)(g(x)) \cdot (\frac{d}{dx} g)(x).$$

I want to emphasize in words what the chain rule says: If you want to compute the derivative of $f \circ g$ at x, then you must compute two things:

1. The derivative of f **at g(x)**, and
2. The derivative of g at x.

The product of these two numbers gives the derivative of $f \circ g$ at x.

Example 9.0.2. Find the derivative of $\sin(x^2+5)$.

Solution. We must first recognize that we behold a composition of two functions: On the outside is sin, while the inside is x^2+5. Hence we can use the chain rule.

$$\frac{d}{dx}(\sin(x^2+5)) = (\frac{d}{dx}\sin)(x^2+5)\cdot\frac{d}{dx}(x^2+5).$$

This is a product of two factors: The first factor, on the left, is the derivative of sin, evaluated at x^2+5. The second factor, on the right, is the derivative of x^2+5 (evaluated at x).

Because we know $\frac{d}{dx}\sin = cos$, and that $\frac{d}{dx}(x^2+5) = 2x$, we conclude:

$$\frac{d}{dx}(\sin(x^2+5)) = \cos(x^2+5)\cdot 2x.$$

Or, in more palatable notation,

$$\frac{d}{dx}(\sin(x^2+5)) = 2x\cos(x^2+5).$$

Exercise 9.0.3. Find the derivatives of the following functions:

1. $(\cos(x)+\sin(x))^3$
2. $\cos(\sin(x))$
3. $\cos(2x^4)$.

Exercise 9.0.4. We do not yet know how to take derivatives of a function like $h(x) = x^{1/3}$. *However*, we do know that if $g(x) = x^3$, then $g(h(x)) = x$.

Using this, and the chain rule, can you find a formula for $h'(x)$? That is, can you compute the derivative of $x^{1/3}$?

9.1 Preparation for Lecture 10: Quotient Rule

In preparing for Lecture 10, you are going to learn about the *quotient rule*.

First, let's suppose $g(x)$ is a function whose derivative we understand. Let's try to figure out the derivative of the function

$$\frac{1}{g(x)}.$$

Example 9.1.1. We know the derivative of $x^3 + 3$. Can we compute the derivative of

$$\frac{1}{x^3+3}?$$

Here is a fact:

Lemma 9.1.2. Whenever $g(x) \neq 0$ and g is differentiable at x, we have:

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-\frac{dg}{dx}(x)}{g(x)^2}.$$

Put another way,

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}.$$

Here is a proof of this fact:

Proof. Let's begin by noticing that

$$1 = g(x) \cdot \frac{1}{g(x)} \qquad \text{(whenever } g(x) \neq 0\text{)}.$$

Because the function on the right is equal to the function on the left (they are both constant functions), their derivatives will be equal. Taking the derivatives of both sides, we have:

$$0 = \frac{d}{dx}(g(x) \cdot \frac{1}{g(x)}).$$

Using the product rule on the righthand side, we find:

$$0 = g'(x) \cdot \frac{1}{g(x)} + g(x) \cdot \frac{d}{dx}\frac{1}{g(x)}.$$

Moving a term over to the left, we find

$$-g'(x) \cdot \frac{1}{g(x)} = g(x) \cdot \frac{d}{dx}\frac{1}{g(x)}.$$

Dividing by $g(x)$ and simplifying, we find:

$$\frac{-g'(x)}{g(x)^2} = \frac{d}{dx}\frac{1}{g(x)} \qquad \text{when } g(x) \neq 0.$$

This proves the lemma! □

Example 9.1.3. Now let's try tackling the question from Example 9.1.1. Let $g(x) = x^3 + 3$. Then we know $g' = 3x^2$. So the formula from the Lemma tells us:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{g(x)}\right) &= \frac{-g'(x)}{g(x)^2} && (9.1) \\ &= \frac{-(3x^2)}{(x^3+3)^2} && (9.2) \\ &= \frac{-3x^2}{(x^3+3)^2}. && (9.3) \end{aligned}$$

I won't simplify this fraction any further; though you could multiply out the bottom if you like.

Now, we will finally know how to take derivatives of quotients:

Theorem 9.1.4 (The quotient rule.)**.** Whenever f and g are differentiable at x, and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}.$$

Put another way,

$$\frac{d}{dx}\left(\frac{f}{g}\right)(x) = \frac{\frac{df}{dx}(x)g(x) - \frac{dg}{dx}(x)f(x)}{g(x)^2}.$$

In case you're curious, here is the proof of the quotient rule:

Proof of the quotient rule. Let's note that

$$\frac{f}{g} = f \cdot \frac{1}{g}.$$

So we can compute

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right) && (9.4)\\
&= (\frac{d}{dx}f)(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{d}{dx}\left(\frac{1}{g(x)}\right) && (9.5)\\
&= \frac{\frac{df}{dx}(x)}{g(x)} + f(x)\cdot\left(\frac{-\frac{dg}{dx}(x)}{g(x)^2}\right) && (9.6)\\
&= \frac{g(x)\frac{df}{dx}(x)}{g(x)^2} + \frac{-f(x)\frac{dg}{dx}(x)}{g(x)^2} && (9.7)\\
&= \frac{\frac{df}{dx}(x)g(x) - f(x)\frac{dg}{dx}(x)}{g(x)^2}. && (9.8)
\end{aligned}$$

The most important thing to note is that to conclude the equality in (9.6), we used Lemma 9.1.2 from above.

The beginning and the end of this string of inequalities is exactly what the quotient rule says. □

Remark 9.1.5. If the notation in the above proof is unappealing, here is a proof using only the "prime" notation:

$$\begin{aligned}
\left(\frac{f}{g}\right)' &= \left(f\cdot\frac{1}{g}\right)' && (9.9)\\
&= f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)' && (9.10)\\
&= \frac{f'}{g} + f\cdot\left(\frac{-g'}{g^2}\right) && (9.11)\\
&= \frac{gf'}{g^2} + \frac{-fg'}{g^2} && (9.12)\\
&= \frac{f'g - fg'}{g^2}. && (9.13)
\end{aligned}$$

Again, the equality in (9.11), follows from Lemma 9.1.2 from above.

Warning 9.1.6. The hardest part about the quotient rule is remembering the order of things in the numerator:

$$f'g - fg'.$$

Note that the positive term is $f'g$, and the negative term is fg'. Some people prefer to remember the numerator as

$$f'g - g'f,$$

which is the same thing.

Do *not* make the mistake of writing something like $g'f - f'g$ in the numerator. This is the wrong answer.

Example 9.1.7 (The derivative of tangent)**.** Let's compute the derivative of

$$\frac{\sin(x)}{\cos(x)}.$$

(This is known, of course, as the *tangent* function.) Follow along:

$$\begin{aligned}
\left(\frac{\sin}{\cos}\right)' &= \frac{\sin' \cdot \cos - \sin \cdot \cos'}{\cos^2} && (9.14)\\
&= \frac{\cos \cdot \cos - \sin \cdot (-\sin)}{\cos^2} && (9.15)\\
&= \frac{\cos^2 + \sin^2}{\cos^2} && (9.16)\\
&= \frac{1}{\cos^2} && (9.17)\\
&= \sec^2 . && (9.18)
\end{aligned}$$

The first equality is using the quotient rule. The equality (9.17) follows from the identity $\sin^2 + \cos^2 = 1$. (You will be expected to know this identity; it's from trigonometry!) The last equality (9.18) follows from the definition of secant: $\sec = 1/\cos$.

We have proven:

$$\frac{d}{dx} \tan = \sec .$$

Equivalently,

$$\tan'(x) = \sec(x).$$

Example 9.1.8. Find the derivative of

$$\frac{x^2-3}{x^3+1}.$$

We use the quotient rule:

$$\left(\frac{x^2-3}{x^3+1}\right)' = \frac{(x^2-3)'\cdot(x^3+1)-(x^2-3)\cdot(x^3+1)'}{(x^3+1)^2} \tag{9.19}$$

$$= \frac{2x\cdot(x^3+1)-(x^2-3)\cdot 3x^2}{(x^3+1)^2} \tag{9.20}$$

$$= \frac{2x^4+2x-3x^4+9x^2}{(x^3+1)^2} \tag{9.21}$$

$$= \frac{-x^4+9x^2+2x}{(x^3+1)^2}. \tag{9.22}$$

For next class, I expect you to be able to do all of the following:

Exercise 9.1.9. Using the quotient rule, find the derivatives of the following functions:

(a) $\frac{1}{x^2}$

(b) $\frac{x-1}{x^2+3}$

(c) $\frac{\sin(x)}{\cos(x)}$

Exercise 9.1.10. Find the slope of the tangent line at $x=2$ of the function

$$f(x) = \frac{x-1}{x+1}.$$