Lecture 10

Derivatives of exp and ln

10.1 Review

We have learned almost all the important laws of derivatives!

Rule	For limits	For derivatives
Constants	$\lim_{x\to a} C = C$	$\frac{d}{dx}(C) = 0.$
Scaling	$\lim_{x\to a} mf(x) = m \lim_{x\to a} f(x)$	$(mf)' = m \cdot f'$
Sums	$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} + g(x)$	$(f+g)' = f' + g'$
Powers	$\lim_{x\to a} x^n = a^n$	$(x^n)' = nx^{n-1}.$
Products	$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$	$(fg)' = f'g + fg'.$
Quotients	$\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$	$(f/g)' = (f'g - g'f)/(g^2)$
Composition	$\lim_{x\to a}(f(g(x)) = f(\lim_{x\to a} g(x)).$	$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Remark 10.1.1. The "product rule" for the derivative is also called the Leibniz rule. The "composition rule" is never called the composition rule; it is actually called the "chain rule."

Warning 10.1.2. All the rules have *hypotheses* that need to be satisfied before applying them. For limit rules, you must know that the limits on the righthand side of the equalities exist already. (And, to apply the quotient rule, you must know that the limit of the denominator is not zero.)

For derivative rules, you must know that the derivatives on the righthand side of the equalities exist already.(And, to apply the quotient rule, you must know that the denominator is not zero.)

We have also seen *examples* of taking derivatives; in fact:

Expectation 10.1.3. From hereon, you are expected to know how to take the derivatives of:

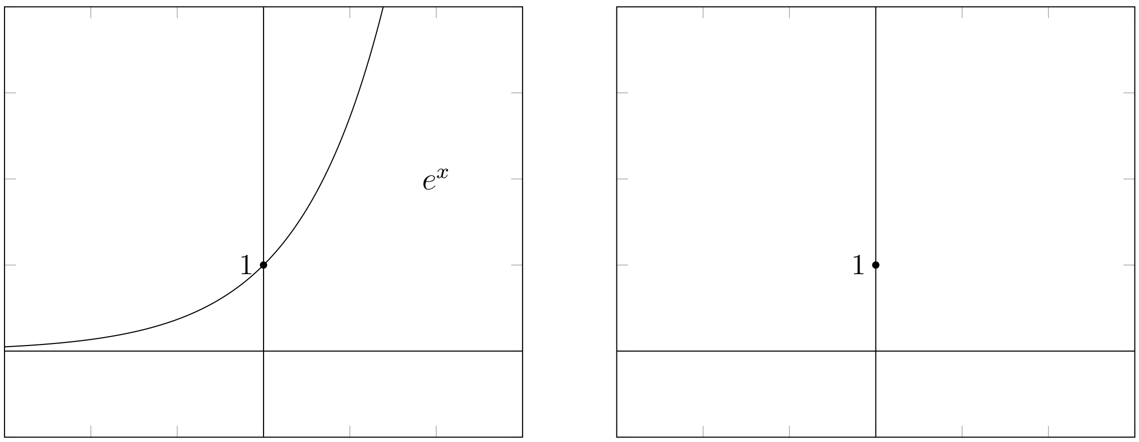
1. Polynomials, like $f(x) = x^5 + 4x^3 + 2x - 3$, and
2. Trigonometric functions like $\sin(x), \cos(x), \tan(x), \sec(x)$.

Here is the general outline of how to do this:

1. To take the derivative of a polynomial, you apply the power law to each term in the polynomial, and then invoke the addition law.
2. To take the derivative of a trigonometric function, you must have memorized the derivatives of sin and cos; and then you write the trig function in terms of sin and cos, and you apply laws like the quotient law.

10.2 Exponentials

Exercise 10.2.1. Below on the left is the graph of $f(x) = e^x$.



Let me tell you the following fact: The derivative of e^x at $x = 0$ is 1. (In fact, the value of e^x at $x = 0$ is 1 also.)

(a) Based on this, draw the derivative of e^x on the right.

(b) How does your drawing compare to the graph of e^x?

In fact, we have the following theorem:

Theorem 10.2.2 (Derivative of e^x)**.** The derivative of e^x is itself. That is,

$$(e^x)' = e^x.$$

Put another way,

$$\frac{d}{dx}(e^x) = e^x.$$

How cool is that? There's a function that is its *own* derivative!

Example 10.2.3. Let's find the derivative of e^{3x}. We have

$$\frac{d}{dx}(e^{3x}) = \frac{d}{dx}(3x) \cdot \frac{d(e^x)}{dx}(3x) \tag{10.1}$$
$$= 3 \cdot e^{3x}. \tag{10.2}$$

We have used the chain rule in the first line. If you're confused by it, it may be worthwhile to write this out step-by-step. Let's let $f(x) = e^x$ and $g(x) = 3x$. Then $e^{3x} = f(g(x))$. Thus

$$\frac{d}{dx}(e^{3x}) = \frac{d}{dx}f(g(x)) \tag{10.3}$$
$$= f'(g(x)) \cdot g'(x) \tag{10.4}$$
$$= f'(3x) \cdot g'(x). \tag{10.5}$$

(The second equality is due to the chain rule.) But we know that $f'(x) = e^x$ by Theorem 10.2.2, and we know $g'(x) = 3$ from previous lectures. Hence we can continue:

$$f'(3x) \cdot g'(x) = e^{3x} \cdot 3 = 3e^{3x}.$$

Exercise 10.2.4. Fix a real number B. Prove that the derivative of

$$f(x) = e^{Bx}$$

equals

$$Bf(x).$$

More generally, if you have another real number A, let

$$g(x) = Ae^{Bx}.$$

(For example, if you choose $A = 3$ and $B = 5$, you would have $3e^{5x}$. The previous example is when $A = 1$.) Prove that

$$g'(x) = Bg(x).$$

Application 10.2.5. This kind of behavior is incredibly important for *modeling*. For example, how fast is a population growing? In ideal circumstances, the more individuals there are in a population, the faster we expect the population to grow. Better yet, we might expect that the rate of population growth is *proportional* to the population itself! (Note that "being proportional to" is a far more precise relationship than "the bigger the population, the faster the growth".)

That's exactly what Exercise 10.2.6 tells us about $g(x) = Ae^{Bx}$. We see that g' is proportional to g (with proportional constant B). So for example, x could model time, while $g(x)$ could model the population at time x.

By the way, why might $g(x)$ be a *bad* model for population growth? For what kinds of situations might $g(x)$ be a *good model*? In those situations, what might A and B represent?

Exercise 10.2.6. Find the derivative of $f(x) = 5^x$. Hints: Remember that $5 = e^{\ln 5}$, remember the basic rules for dealing with exponents, and use the chain rule.

Exercise 10.2.7. Your friend is excited about the idea that $f(x)$ could equal $f'(x)$ and looks for more examples that looks like e^x. They try $f(x) = 5^x$, and are disappointed that $f'(x)$ does not equal $f(x)$.

Is it possible to find any number k—other than e—so that if $f(x) = k^x$, then $f'(x) = f(x)$?

Remark 10.2.8. Isn't e special?

Exercise 10.2.9. Now that you know the derivative of $f(x) = e^x$, can you figure out the derivative of $g(x) = \ln x$?

Hint: What is $f \circ g$? What if you try computing $(f \circ g)'$?

10.3 Review of right inverses

Let f be a function. Here's a question: Given a value of f, can we always determine which x it came from?

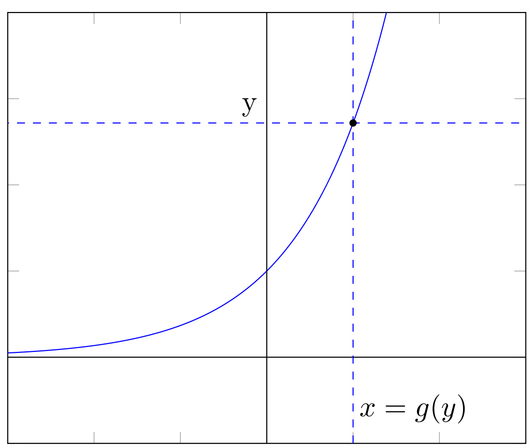
Example 10.3.1. Here are some examples:

1. If $f(x) = 3x$, and if someone tells you that f takes the value 12, you know exactly where: x must equal 4. In fact, in general, if f takes value y, you know the original x is $y/3$.

2. If $f(x) = 2^x$, and if someone tells you that f takes the value 8, you know exactly where: x must equal 3. In fact, in general, if f takes value y, you know f does so at $\log_2 y$.

3. If $f(x) = x^2$, and if someone tells you that f takes the value 4, you *don't* know exactly where: x could equal 2 or -2. However, *if* you restrict yourself to looking only for positive values of x, then if f takes value y, you know that the original x is $\sqrt{y}$.

Below is a visual way to think about this process. Drawn is the graph of f. Given a value y, can you figure out which value of x satisfies the equation $f(x) = y$? If so, that means that the coordinate x now becomes a function of y—you input y, and you output x—and we can call this function g.



Warning 10.3.2. While we were diligent about drawing g as a function of y before, from now on, we must now be comfortable realizing that letters

are just letters, and we don't care if g takes inputs to be symbols that look like "x," or symbols that look like "y"; that is, g will often be treated as a function of x, too.

Definition 10.3.3. Let g be a function. We say that a function f is a *left inverse* to g if

$$(f \circ g)(x) = x.$$

Put another way, f "remembers" the original value x that outputted $g(x)$.

We also say that g is a *right inverse* to f. Put another way, g "knows" that if $f(\text{something}) = x$, then something $= g(x)$.

10.4 Derivatives of right inverses

It turns out that if we know the derivatives of a function f, then—if f has a right inverse g—we can figure out the derivatives of the right inverse g.

Lemma 10.4.1. Let f be a function, and suppose that g is a right inverse to f, defined on some open interval containing x. Suppose also that f is differentiable at $g(x)$, and that $f'(g(x)) \neq 0$. Then

$$g'(x) = \frac{1}{f'(g(x))}. \tag{10.6}$$

That is, the derivative of g at x is computed by dividing 1 by the derivative of f *at* $g(x)$.

Proof. Let's look at the following string of equalities:

$$\begin{aligned} 1 &= (x)' && (10.7)\\ &= (f \circ g)' && (10.8)\\ &= (f' \circ g) \cdot g'. && (10.9) \end{aligned}$$

The first equality is our knowledge of the derivative of the function x. The next equality is using the hypothesis that g is a right inverse to f, so that $f \circ g = x$. The next step is the chain rule.

In total, what this string of equalities says is that the function on the righthand side is equal to the (constant!) function on the lefthand side. So let's evaluate at some point x. We have

$$1 = f'(g(x)) \cdot g'(x).$$

We can divide both sides by $f'(g(x))$ so long as this number isn't zero; so we find:

$$\frac{1}{f'(g(x))} = g'(x) \qquad \text{when } f'(g(x)) \neq 0.$$

This is what we wanted. □

Example 10.4.2. $g = \ln(x)$ is a right inverse to $f(x) = e^x$. This is because

$$f \circ g(x) = e^{\ln x} = x.$$

We know the derivative of $f(x) = e^x$, so we can use the lemma to find the derivative of $g(x) = \ln(x)$! Let's try:

$$
\begin{aligned}
(\ln(x))' &= g'(x) && (10.10)\\
&= \frac{1}{f'(g(x))} && (10.11)\\
&= \frac{1}{e^{g(x)}} && (10.12)\\
&= \frac{1}{e^{\ln(x)}} && (10.13)\\
&= \frac{1}{x}. && (10.14)
\end{aligned}
$$

. The first equality is by definition of g. The next equality is using Lemma 10.4.1. The rest is just plugging in our knowledge of f' and ln.

10.5 The derivative of natural log

The example from the last page is important, so let's record this as a theorem. (You will be expected to know this:)

Theorem 10.5.1 (The derivative of ln)**.** The derivative of ln is "one over x." That is,

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

10.6 Preparation for Lecture 11

For next lecture, you will practice taking "second derivatives," and knowing when they are positive or negative.

Definition 10.6.1. The second derivative of f is *the derivative of the derivative*[1] of f. We denote the second derivative by

$$f'', \qquad \text{or} \qquad \frac{d}{dx}(\frac{d}{dx}f), \qquad \text{or} \qquad \frac{d^2}{dx^2}f, \qquad \text{or} \qquad \frac{d^2f}{dx^2}. \tag{10.15}$$

Example 10.6.2. Let $f(x) = 3x^2 + x - 7$. Then the (first) derivative of f is

$$f'(x) = 6x + 1.$$

If we take the derivative of $f'(x)$, we end up with the second derivative of f:

$$f''(x) = 6.$$

Example 10.6.3. Here are more examples of functions and their second derivatives. You should verify these examples:

- If $f(x) = \sin(x)$, then $f''(x) = -\sin(x)$.
- If $f(x) = e^x$, then $f''(x) = e^x$.
- If $f(x) = e^{5x}$, then $f''(x) = 25e^{5x}$.
- If $f(x) = x^3 - 5x^2$, then $f''(x) = 6x - 10$.

Example 10.6.4. Let's find the second derivative of $f(x) = \ln(x)$. As defined above, we just need need to take the derivative twice. Let's take the first derivative:

$$f'(x) = \frac{1}{x}.$$

(This is something we learned in class.) Now let's take another derivative—for example, by using the quotient rule—to find

$$f''(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}.$$

That is, the second derivative of $\ln x$ is $-1/(x^2)$.

[1] Yes, there are two appearances of the word "derivative"; this is not a typo.

If you know how to take derivatives, you know how to take second derivatives. So you see how our skills are building on each other—make sure you practice taking derivatives!

Example 10.6.5. Let $f(x) = x^2 - 2$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 2x$$

so

$$f''(x) = 2.$$

So the second derivative is always 2, meaning the second derivative is positive *everywhere*.

Example 10.6.6. Let $f(x) = x^3 - 3x^2 + 3$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 3x^2 - 6x$$

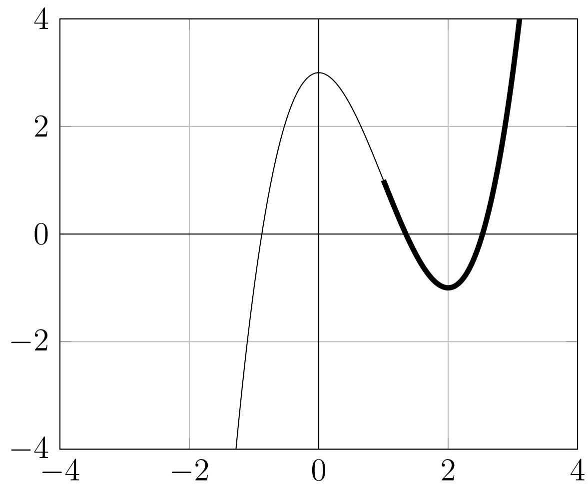
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = 6x - 6.$$

So the second derivative is positive when $6x - 6$ is positive. This happens exactly when $6x > 6$—that is, when $x > 1$.

As a bonus: The second derivative is negative when $6x < 6$—that is, when $x < 1$.

Below is a graph of $f(x)$, and I have shaded in **bold** the part of the graph where the second derivative is positive:



Example 10.6.7. Let $f(x) = x^4 - 24x^2 + 50$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 4x^3 - 48x$$

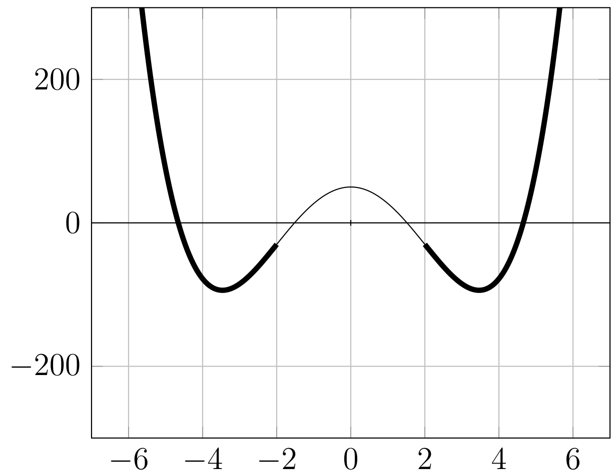
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = 12x^2 - 48.$$

So the second derivative is positive when $12x^2 - 48$ is positive. This happens exactly when $12x^2 > 48$—that is, when $x^2 > 4$. But $x^2 > 4$ exactly when $x < -2$ or $x > 2$.

As a bonus: The second derivative is negative when $x^2 < 4$—that is, when x is between -2 and 2.

Below is a graph of $f(x)$, and I have shaded in **bold** the part of the graph where the second derivative is positive:



Example 10.6.8. Let $f(x) = 3\sin(x)$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 3\cos(x)$$

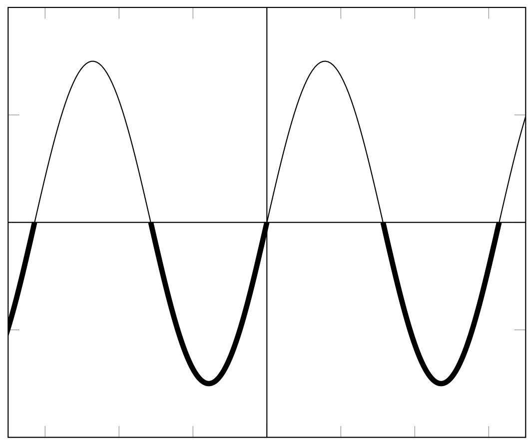
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = -3\sin(x)$$

So the second derivative is positive when $-3\sin(x)$ is positive. This happens exactly when $\sin(x)$ is negative. And based on our trigonometry knowledge from precalculus, we know that this happens when

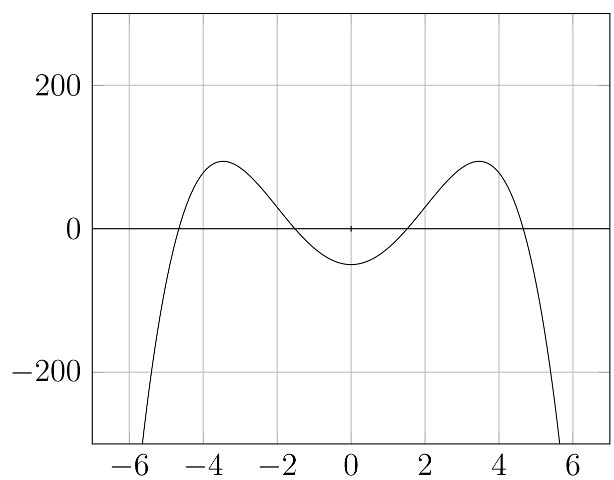
- x is between π and 2π,
- x is between 3π and 4π,
- x is between $-\pi$ and 0,
- x is between -3π and $-\pi$,
- $\ldots$.

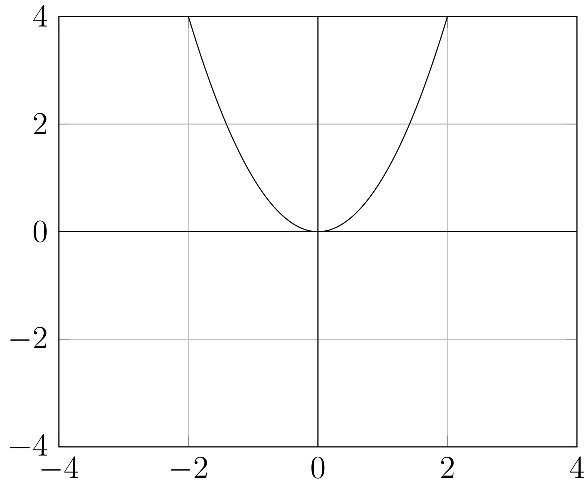
Below is a graph of $f(x)$. I have shaded in **bold** the part of the graph where the second derivative is positive:



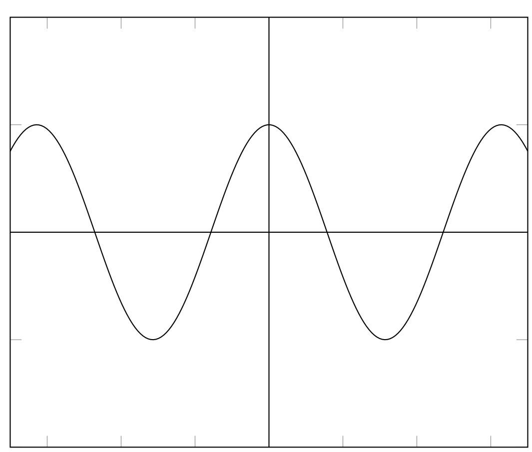
For next class, I expect you to be able to do the following: For each of the functions $f(x)$ below, (i) State *where* the function has a *positive* second derivative, and (ii) Shade in **bold** where the graph of the function has positive second derivative. (You will be provided the graph of $f(x)$.)

(a) $f(x) = -x^4 + 24x^2 - 50.$

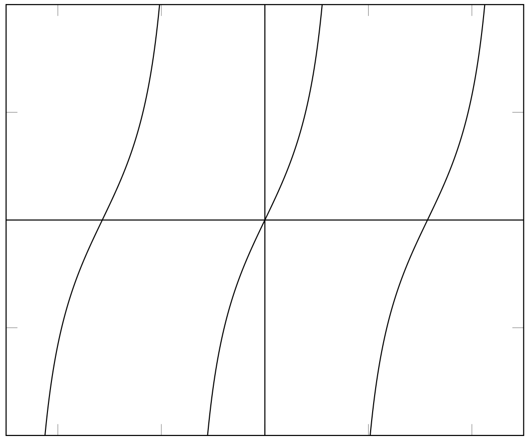




(b) $f(x) = x^2$.



(c) $f(x) = \cos(x)$.



(d) $f(x) = \tan(x)$.