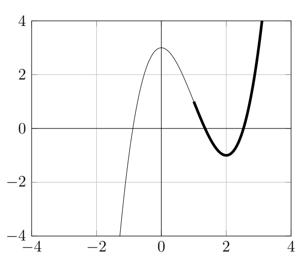
Lecture 11

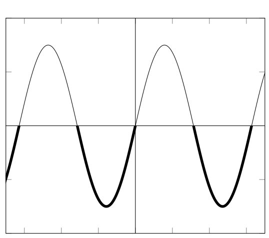
Concavity, extrema, critical points

You have seen examples of graphs with positive second derivative. Here are some examples, with the positive-second-derivative regions shaded in *bold*:

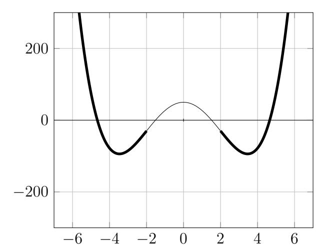
1. $f(x) = x^3 - 3x^2 + 3$:



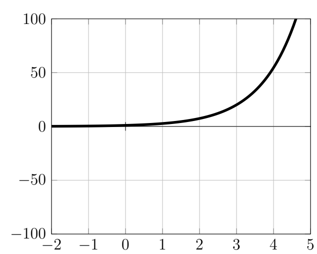
2. $f(x) = 3\sin(x)$:



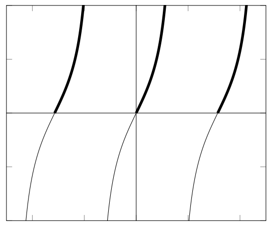
3. $f(x) = x^4 - 24x^2 + 50$:



4. $f(x) = e^x$:



5. $f(x) = \tan(x)$:



11.1 Concavity

The point I want to make with these pictures is that *the value of the second derivative gives us some idea of what the graph looks like.* (Though not a complete picture.)

Intuition: On the regions where the second derivative is positive, the graph of f looks like a *portion* of an "upright bowl." Some students have described this as "opening upward" as well.

Conversely, when the second derivative is negative, the graph of f looks like a portion of an "upside-down bowl." But we have technical names, too. From now on, you are expected to know the following terminology:

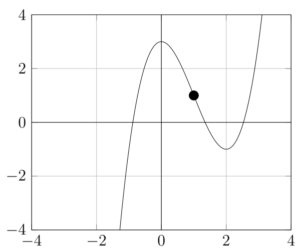
Definition 11.1.1 (Concavity)**.** We say that f is *concave up* at x if $f''(x) > 0$. We say that f is *concave down* at x if $f''(x) < 0$.

11.2 Inflection points

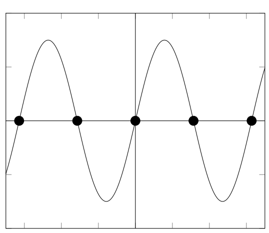
Definition 11.2.1. If $f''(x) = 0$, and the concavity of f *changes* at x, we say that x is an *inflection point*.

Example 11.2.2. Here are some examples of functions and their graphs, with their inflection points labeled.

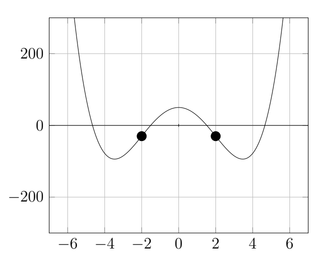
1. $f(x) = x^3 - 3x^2 + 3$:



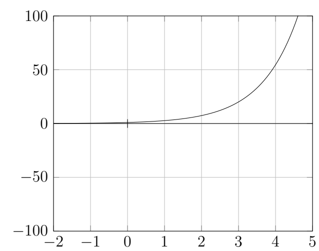
2. $f(x) = 3\sin(x)$:



3. $f(x) = x^4 - 24x^2 + 50$:

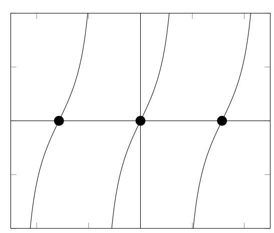


4. $f(x) = e^x$:

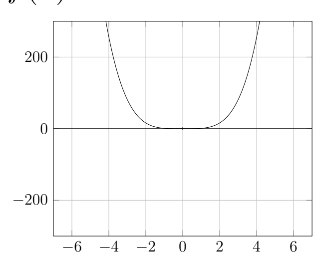


(No inflection points.)

5. $f(x) = \tan(x)$:



6. $f(x) = x^4$:



(No inflection points, even though $f''(x) = 0$ at $x = 0$.)

Expectation 11.2.3. Based on looking at a graph, you are expected to be able to identify inflection points—an inflection point is a place at which a function switches concavity (from up to down, or from down to up).

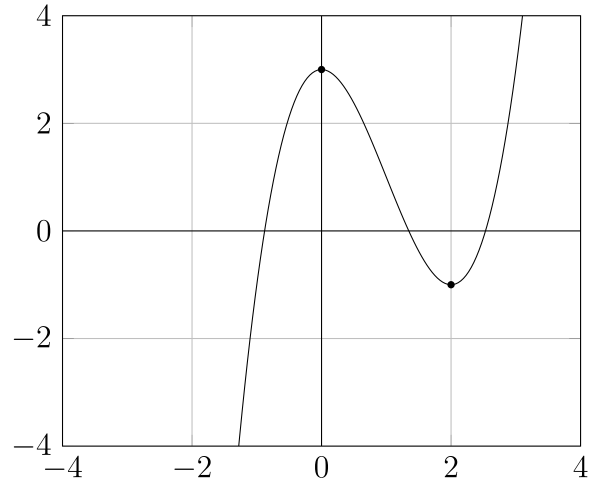
Exercise 11.2.4. Now we're going to try to learn something about a function by knowing its derivative and second derivative. You can hunt for examples on the previous pages of this packet. Or, you can try understanding $f(x) = x^2$ and $f(x) = -x^2$.

1. Can you find an example of a function f, and a point x, where $f'(x) = 0$ and f is concave up at x? What does the function f look like near x? How does the value of f at x compare to the value of f at nearby points?

2. Can you find an example of a function f, and a point x, where $f'(x) = 0$ and f is concave *down* at x? What does the function f look like near x? How does the value of f at x compare to the value of f at nearby points?

11.3 Local extrema

Let's study the example of $f(x) = x^3 - 3x^2 + 3$:



Just by looking at the graph, we can see the two points where the derivative of f is zero (i.e., the two points where the tangent lines are horizontal). They roughly occur at $x = 0$ and $x = 2$. (And you can prove that they *exactly* occur there if you do out the math—that is, if you solve the equation $f'(x) = 0$ for x.)

We see that at $x = 0$, the function is concave down. Moreover, it looks like $f(0)$ is the *biggest* value that f achieves near $x = 0$. We will call such a point a **local maximum**. (That is, $x = 0$ is a local maximum.)

And at $x = 2$, we see that the function is concave up. Moreover, it looks like $f(2)$ is the *smallest* value that f achieves near $x = 2$. We call such a point a **local minimum** (so $x = 2$ is a local minimum). A point is called a local extremum (the plural is "local extrema") if it is either a local maximum or a local minimum.

Your intuition might tell you that wherever there is a local maximum or a local minimum, the graph should have a "trough" or a "crest." In particular, the derivative should be zero there! This is true so long as the function is differentiable:

Theorem 11.3.1. If f is a differentiable function, and if x is a local minimum or a local maximum, then $f'(x) = 0$.

Warning 11.3.2. These minima and maxima are called "local." This is because if x is a local minimum, it may not be true that $f(x)$ is the "minimum" value that f can take!

In the example above of $f(x) = x^3 - 3x^2 + 3$, we see that $f(x)$ can take as negative a value as it wants, so f has no "absolute minimum." Likewise, $f(x)$ can take as positive value as it wants, so f has no "absolute maximum." It only has a "local" minimum at $x = 2$, where the value of $f(2)$ is smaller than the value at all *neighboring* points (i.e., all nearby points).

11.4 Critical points

So it will be important for us to find x for which f' vanishes. Such special points have a name:

Definition 11.4.1. Let f be a function. If f is differentiable at x, we say that x is a *critical point* of f if $f'(x) = 0$.

Example 11.4.2. If $f(x) = 5$, every point is a critical point.

If $f(x) = 3x$, f has no critical points.

If $f(x) = x^2$, $x = 0$ is a critical point.

In fact, zero is a critical point for $f(x) = x^3$ and for $f(x) = x^4$, and so forth.

Warning 11.4.3. Not all critical points are local extrema. (For example, look at the critical point of $f(x) = x^3$.)

Warning 11.4.4. If f is not differentiable, not all local extrema are critical points. Consider the example of $f(x) = |x|$. This has a minimum at $x = 0$, but f does not have a derivative there (as we have seen before).

11.5 The second derivative test

The following is called the **second derivative test** for finding local maxima and local minima. You in fact discovered it when thinking about Exercise 11.2.4.

Theorem 11.5.1 (The second derivative test)**.** Suppose that $f'(x) = 0$ and $f''(x) > 0$. Then f has a *local minimum* at x.

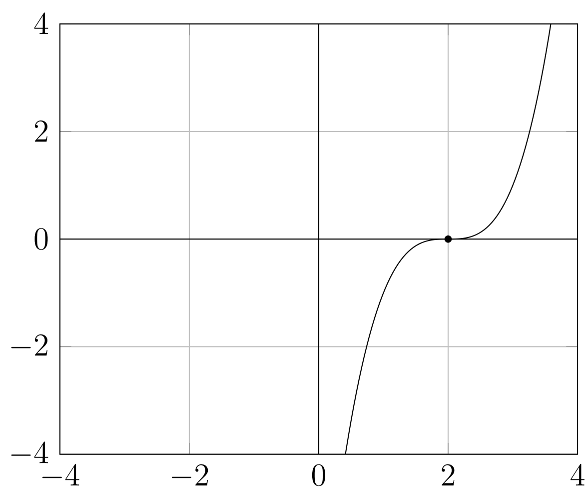
Suppose that $f'(x) = 0$ and $f''(x) < 0$. Then f has a *local maximum* at x.

This helps us draw f: We know that f looks like a hump/hilltop/crest where f has a local maximum. And we know that f looks like a bowl/trough/nadir where f has a local minimum.

11.6 The second derivative test can be inconclusive

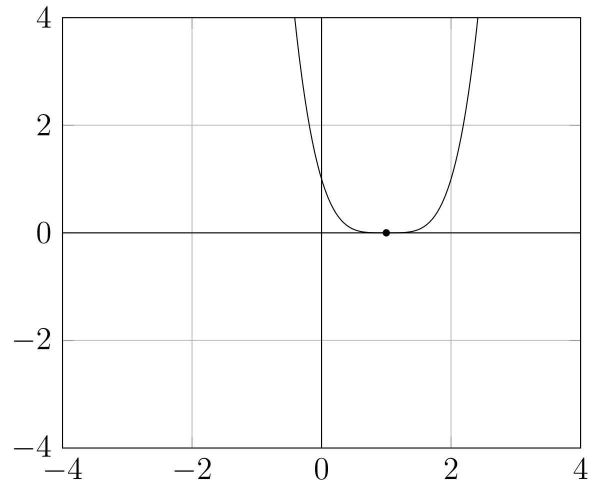
If $f'(x) = 0$ and $f''(x) = 0$, we *do not know* whether we have a local maximum or minimum (or neither)! Here are two examples:

Example 11.6.1. Consider $f(x) = (x-2)^3$. Then—check this!—$f'(2) = 0$ and $f''(2) = 0$. Below is a graph of $f(x)$:



This is a strange example, but it is a great one. As you can see, the graph does have "flat" tangent line at $x = 2$, but $x = 2$ is neither a local maximum nor a local minimum—I can immediately get larger than $f(2) = 0$ by moving right, or immediately get smaller than $f(2) = 0$ by moving left.

Example 11.6.2. Here is the example of $f(x) = (x-1)^4$. We can check easily that $f'(1) = 0$ and $f''(1) = 0$.



As we can see from the picture, we have a *local minimum* at $x = 1$.

The conclusion from the above two examples is: If the hypotheses of the second derivative test are not met, we have to do more work to determine whether we have a local minimum or maximum.

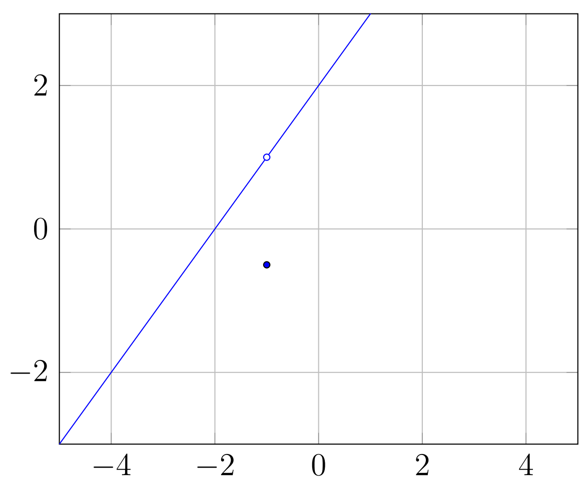
11.7 Preparation for Lecture 12

11.7.1 Finding local extrema

The second derivative test is the most useful tool we have for finding local extrema. You must be cautious that the test is *sometimes inconclusive*, and we won't learn how to deal with those inconclusive cases in this course (things can become complicated on a case-by-case basis).

How to find local extrema of f graphically: When f is continuous, identify where the function seems to "peak" or "rebound;" equivalently, find the "crests" and the "troughs."

Warning 11.7.1. When f is not continuous, there are annoying examples like the following:



Here, $x = -1$ is a local minimum because $f(-1) = -0.5$ is smaller than f at any horizontal coordinate nearby $x = -1$. We will largely ignore pathological examples like this because they don't arise very often when we care about local extrema.

How to find local extrema of f algebraically: This is useful because sometimes, we don't know what the graph of f looks like.

1. Identify the critical points of f. (This means to find the values of x for which $f'(x) = 0$.)

2. Compute f'', and compute $f''(x)$ at the critical points. (This means to find f'', then plug in the values of x from part 1.)

3. Apply the second derivative test: If $f'' > 0$ at a critical point, that critical point is a local minimum. If $f'' < 0$, it is a local maximum.

4. If $f'' = 0$ at some critical point, the second derivative test is inconclusive. If the problem needs you to, you must find some clever way to conclude whether you have discovered a local maximum, minimum, or neither. Or, you can throw your hands up and say "The second derivative test is inconclusive."

Example 11.7.2. Find all local extrema of $f(x) = (x-3)e^x$.

First, let's note that f is differentiable. (This follows from the Leibniz rule—the product of two differentiable functions is differentiable, and each factor in $f(x)$ is differentiable.[1]) This means any local extrema will be critical points.[2] So let's find the critical points. We must first compute f', and we find—using the product rule:

$$f'(x) = (x-3)'e^x + (x-3)(e^x)' \tag{11.1}$$
$$= e^x + (x-3)(e^x) \tag{11.2}$$
$$= (x-2)e^x. \tag{11.3}$$

We note that e^x never equals zero, so the only critical point is at $x = 2$. (Make sure you understand this!)

Now we apply the second derivative test. To do so, we must compute f'':

$$f''(x) = (x-2)'e^x + (x-2)(e^x)' \tag{11.4}$$
$$= e^x + (x-2)e^x \tag{11.5}$$
$$= (x-1)e^x. \tag{11.6}$$

At $x = 2$, we find:

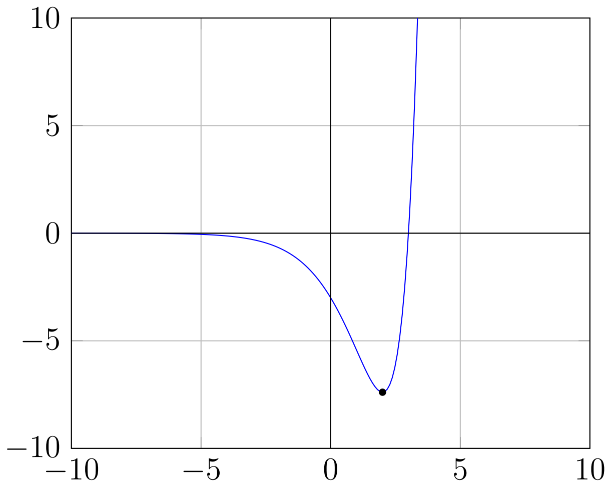
$$f''(2) = (2-1)e^2 \tag{11.7}$$
$$= e^2 \tag{11.8}$$

which is positive. Thus, f' has a *local minimum* at $x = 2$. And (because there are no other critical points) this is the only local extremum of f'.

[1] The two factors are $x - 3$ and e^x

[2] Theorem **??**.

Here is a graph of $f(x) = (x-3)e^x$ to confirm this result:



The local minimum is labeled with a black dot.

Example 11.7.3. Let $f(x) = x^6 - 3x^4 + 3$. Let's find all the local minima and maxima.

First, we know that f is differentiable because it is a polynomial function. By Theorem **??**, we know that all local extrema (that is, all local maxima and minima) occur at critical points. So our first task is to find all critical points: That is, to find all points x such that $f'(x) = 0$.

To accomplish this, we need to compute $f'(x)$. I expect by now that you can use the power rule to find:

$$f'(x) = 6x^5 - 12x^3. \tag{11.9}$$

This factors nicely:

$$f'(x) = x^3(6x^2 - 12) = 6x^3(x^2 - 2) = 6x^3(x + \sqrt{2})(x - \sqrt{2}).$$

So there are **three critical points**: $x = 0$ and $x = - \pm \sqrt{2}$.

Now let's see whether these are local maxima, minima, or neither. For this we need to apply the second derivative test (Theorem **??**). This means we have to compute $f''(x)$! We find:

$$f''(x) = 30x^4 - 36x^2. \tag{11.10}$$

Remember, we need to evaluate f'' at the critical points of x to use the second derivative test. So we need to evaluate three numbers:

$$f''(0), \qquad f''(\sqrt{2}), \qquad f''(-\sqrt{2}).$$

We see that $f''(0) = 0$ because :

$$f''(0) = 30(0)^4 - 36(0)^2 = 0 - 0.$$

We also compute:

$$f''(\sqrt{2}) = 30(\sqrt{2})^4 - 36(\sqrt{2})^2 = 120 - 72 = 48.$$

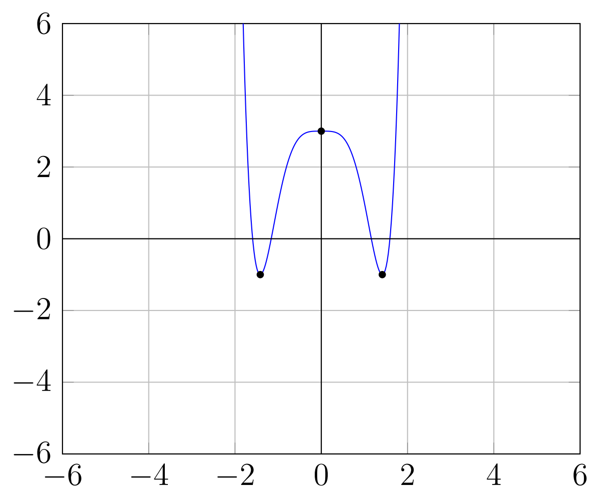
Likewise, $f''(-\sqrt{2}) = 48$.

Here is our conclusion:

- At $x = \pm\sqrt{2}$, we have a critical point and f'' is *positive*. Thus these two points are both local minima (by the second derivative test).
- At $x = 0$, we have a critical point and f'' is *zero*. Thus we must do more work to determine whether $x = 0$ is a local maximum, minimum, or neither.

We will learn how to deal with this particular example at $x = 0$ later in this course. As it turns out, $x = 0$ is a local maximum. If you have free time, you can think about how you might prove that!

The graph is below.



The critical points are labeled in black. (Note that, even though the second derivative test was inconclusive at $x = 0$, the graph clearly suggests that $x = 0$ should be a local maximum!

For next lecture's quiz, I expect you to be able to solve the following: For each of the following functions

1. $f(x) = x^3 e^x$
2. $f(x) = x^3 - 12x$
3. $f(x) = x^6 - 27x^2$

You should be able to

(a) Identify all critical points

(b) Identify local maxima and local minima (using the second derivative test), and

(c) Identify the critical points for which the second derivative test is inconclusive.[3]

[3] Remember, the second derivative test is inconclusive where $f''(x) = 0$.