Lecture 12

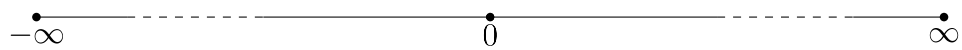
Limits equaling ∞

Warning 12.0.1. We will be using the symbol ∞ a lot. This symbol stands for "infinity." I want you to know that the way we use ∞ in calculus class can be detrimental to understanding the other uses of infinity in mathematics.

There are "infinitely many" integers; this notion of infinity answers the question "how many?". The "how many" notion is subtly, but definitely, different from the notion of ∞ that we'll use in calculus.

12.0.1 Infinity, in our class

Where are ∞ and $-\infty$? This is controversial among some calculus instructors; but in my class, you will treat ∞ and $-\infty$ as though they are "numbers." In fact, you should imagine that I've added two ends to the number line:



So for example, between 0 and ∞ lies *every* positive real number. Between $-\infty$ and 0 lies every negative real number. ∞ is larger than any number; $-\infty$ is lesser than any number.

Remark 12.0.2. This should give you some idea for what it means to *approach* infinity. It means that, for any point T on the real line, you eventually surpass and stay larger than T. Likewise, for you to approach $-\infty$ means that, for any number T on the real line, you eventually become more negative than, and stay more negative than, T.

12.0.2 Arithmetic with ∞ and $-\infty$

Of course, you should be able to add/subtract/multiply/divide numbers. Here are the basic rules you need to remember; they are what you would have guessed. (Below, remember that ∞ and $-\infty$ are *NOT* real numbers.)

- Addition and multiplication are still commutative.

Here are the rules involving addition and subtraction:

- If x is a real number, $x + \infty = \infty$ and $x + (-\infty) = (-\infty)$.[1]
- $\infty + \infty = \infty$ and $(-\infty) + (-\infty) = (-\infty)$.

When taking products and quotients, we must never involve zero with $\pm\infty$:

- If x is a *positive* real number, $x \times \infty = \infty$ and $x \times (-\infty) = (-\infty)$.
- If x is a *negative* real number, $x \times \infty = -\infty$ and $x \times (-\infty) = \infty$.
- If x is a *positive* real number, $\infty/x = \infty$ and $(-\infty)/x = (-\infty)$.

[1] In particular, $\infty - x = \infty$ and $(-\infty) - x = (-\infty)$.

- If x is a *negative* real number, $\infty/x = -\infty$ and $(-\infty)/x = \infty$.
- If x is a real number with $x \neq 0$, then $x/\infty = 0$ and $x/(-\infty) = 0$.
- $\infty \times \infty = \infty$ and $\infty \times (-\infty) = (-\infty)$ and $(-\infty) \times (-\infty) = \infty$.

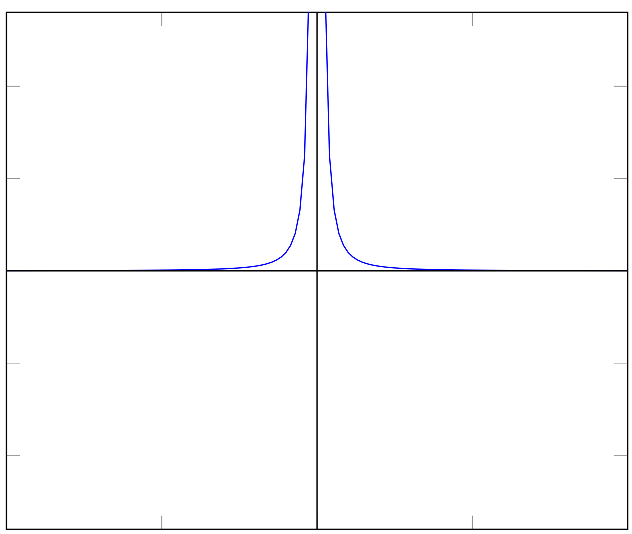
Finally, just as you cannot divide a real number by zero, there are certain operations that are undefined when involving $\pm\infty$:

- ∞ - ∞ and $-\infty + \infty$ are undefined.
- ∞/∞ and $-\infty/\infty$ and $\infty/(-\infty)$ and $(-\infty)/(-\infty)$ are all undefined.
- $0 \times \infty$ and $0 \times (-\infty)$ are undefined.
- $0/\infty$ and $0/(-\infty)$ and $\infty/0$ and $(-\infty)/0$ are undefined.

12.0.3 Limits equaling infinity

We'll talk about limits equaling ∞ via examples.

Example 12.0.3. Consider the function $f(x) = 1/x^2$. Here's a graph of it:



As you know, $f(x) = 1/x^2$ is *not* defined at $x = 0$. However, does f seem to "want" to do something as x approaches zero?

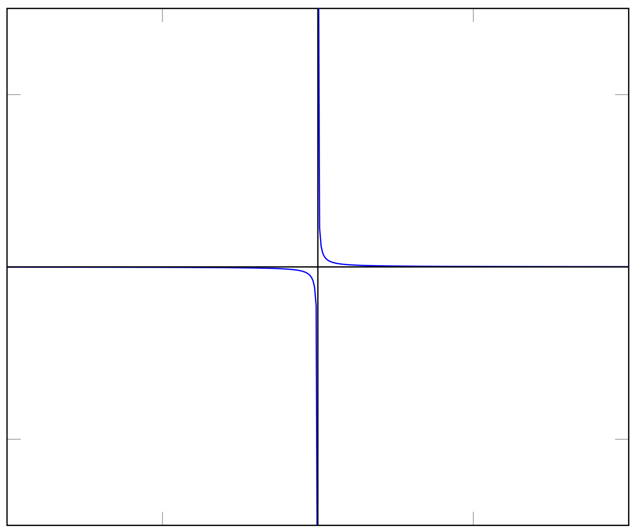
As you see from the graph, f is "spiking" at $x = 0$, and becoming larger and larger. In fact, if there's a height H that you want to surpass, all you have to do is make sure that x is small enough. For every small-enough x, we know $f(x)$ will be larger than H.

Thus, we say:

$$\lim_{x \to 0} f(x) = \infty.$$

This is our first use of ∞ in calculus class!

Example 12.0.4. We can talk about left and right limits equaling ∞, too. Consider the function $f(x) = 1/x$. Here's a graph of it:



As you can see, as we approach the origin from the *right*, the graph of f is spiking upward again. We can talk about this righthand limit:

$$\lim_{x \to 0^+} f(x) = \infty.$$

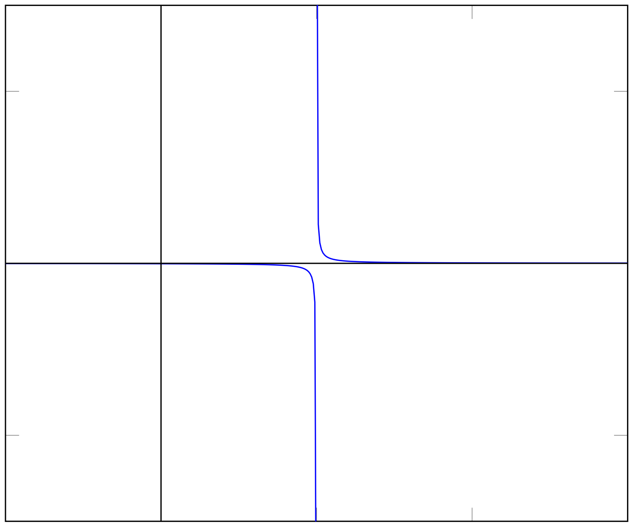
However, as we approach $x = 0$ from the left, the graph of f is spiking *downward*, and f is approaching $-\infty$. Thus, we say:

$$\lim_{x \to 0^-} f(x) = -\infty.$$

Note that the lefthand limit and the righthand limit do *not* agree. Just like limits for real numbers (and not $\pm\infty$), because the two one-sided limits do not agree, we can say:

$$\lim_{x \to 0} f(x) \text{ does not exist.}$$

Example 12.0.5. We can talk about left and right limits equaling ∞, too. Consider the function $f(x) = 1/(x - 0.2)$. Here's a graph of it:



As you can see, as we approach 0.2 from the *right*, the graph of f is spiking upward again. So

$$\lim_{x \to 0.2^+} f(x) = \infty.$$

However, as we approach $x = 0.2$ from the left, the graph of f is spiking *downward*, and f is approaching $-\infty$. Thus, we say:

$$\lim_{x \to 0.2^-} f(x) = -\infty.$$

Note that the lefthand limit and the righthand limit do *not* agree. Just like limits for real numbers (and not $\pm\infty$), because the two one-sided limits do not agree, we can say:

$$\lim_{x \to 0.2} f(x) \text{ does not exist.}$$

There is nothing special about 0.2. In fact, for any real number C, we have that

$$\lim_{x \to C^+} \frac{1}{x - C} = \infty, \qquad \lim_{x \to C^-} \frac{1}{x - C} = \infty, \qquad \lim_{x \to C} \frac{1}{x - C} = \text{does not exist.}$$

12.1 Limit rules, revisited (this time with ∞)

Once you know how to add/multiply/divide/subtract with ∞, and once you know the basic limits, you can begin to compute limits of more complicated functions.

Here are the basic limit laws for infinity; they are like the old ones, just with more caveats about being careful:

1. (**New:** Limits of $1/(x-C)$). For any real number C, we have that
$$\lim_{x\to C^-}\frac{1}{x-C}=-\infty, \qquad \text{and} \qquad \lim_{x\to C^+}\frac{1}{x-C}=\infty.$$
(Make sure to take a look at Example 12.0.5 if you haven't yet.)

2. (Scaling law). When the righthand side is defined, for any real number m, we have
$$\lim_{x\to a} mf(x)=m\lim_{x\to a} f(x).$$
New point of caution: The righthand side is undefined if $m=0$ and if $\lim_{x\to a} f(x)=\pm\infty$.

3. (Puncture law). If $f(x)=g(x)$ away from a, then
$$\lim_{x\to a} f(x)=\lim_{x\to a} g(x).$$

4. (Product law) We have that
$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x).$$
New point of caution: Importantly, the righthand side is not defined when multiplication is not defined—for example, $0\cdot\infty$ is undefined for us. When the righthand side is undefined, you have to *try something different* from the product rule to determine the limit.

5. (Quotient law) We have that
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$
New point of caution: Importantly, the righthand side is not defined when division is not defined—for example, $0/\infty$ is undefined. When the righthand side is undefined, you have to *try something different* from the quotient rule to determine the limit.

Remark 12.1.1. Limit laws also work for one-sided limits! This is a good thing. For example,

$$\lim_{x\to a^+}(f(x)\cdot g(x)) = \lim_{x\to a^+} f(x)\cdot \lim_{x\to a^+} g(x).$$

Example 12.1.2 (This is an example you should memorize the result of). Let's try to establish that

$$\lim_{x\to 0} \frac{1}{x^2} = 0.$$

It suffices to compute both one-sided limits, and to show that they are the same. Here's one:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1}{x^2} &= \lim_{x\to 0^+} (\frac{1}{x} \cdot \frac{1}{x}) && (12.1)\\
&= \lim_{x\to 0^+} \frac{1}{x} \cdot \lim_{x\to 0^+} \frac{1}{x} && (12.2)\\
&= \infty \cdot \infty && (12.3)\\
&= \infty. && (12.4)
\end{aligned}$$

The first line is just algebra. The next line is using the product rule for one-sided limits. Then we are using the fact that we know already the one-sided limits for $1/x$. The last line follows from our rules about arithmetic with ∞.

And here's the other one-sided limit:

$$\begin{aligned}
\lim_{x\to 0^-} \frac{1}{x^2} &= \lim_{x\to 0^-} (\frac{1}{x} \cdot \frac{1}{x}) && (12.5)\\
&= \lim_{x\to 0^-} \frac{1}{x} \cdot \lim_{x\to 0^-} \frac{1}{x} && (12.6)\\
&= (-\infty) \cdot (-\infty) && (12.7)\\
&= \infty. && (12.8)
\end{aligned}$$

In sum, we see that both one-sided limits agree, so $1/x^2$ has a limit at 0. We can conclude:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Example 12.1.3. Let's compute

$$\lim_{x\to 0^+} \frac{1+x}{4x^2}.$$

The fastest approach is to use the product law:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1+x}{4x^2} &= \lim_{x\to 0^+} \frac{1+x}{1} \cdot \lim_{x\to 0^+} \frac{1}{4x^2} && (12.9)\\
&= 1 \cdot \lim_{x\to 0^+} \frac{1}{4x^2} && (12.10)\\
&= 1 \cdot \frac{1}{4} \lim_{x\to 0^+} \frac{1}{x^2} && (12.11)\\
&= \frac{1}{4} \lim_{x\to 0^+} \frac{1}{x^2} && (12.12)\\
&= \frac{1}{4}\infty && (12.13)\\
&= \infty. && (12.14)
\end{aligned}$$

The first line is the product law. (12.11) is computing the limit of $\frac{1+x}{1}$. (This is something you already knew how to do.) (12.12) is the scaling law. The next line is algebra. (12.13) follows from our knowledge of the limit of $1/x^2$ at 0. The last line is arithmetic using ∞.

Here is a different, very tedious approach:

$$
\begin{aligned}
\lim_{x\to 0^+} \frac{1+x}{4x^2} &= \lim_{x\to 0^+} \left(\frac{1}{4x^2} + \frac{x}{4x^2}\right) && (12.15)\\
&= \lim_{x\to 0^+} \frac{1}{4x^2} + \lim_{x\to 0^+} \frac{x}{4x^2} && (12.16)\\
&= 4 \lim_{x\to 0^+} \frac{1}{x^2} + \lim_{x\to 0^+} \frac{x}{4x^2} && (12.17)\\
&= 4 \cdot \infty + \lim_{x\to 0^+} \frac{x}{4x^2} && (12.18)\\
&= \infty + \lim_{x\to 0^+} \frac{x}{4x^2} && (12.19)\\
&= \infty + \lim_{x\to 0^+} \frac{1}{4x} && (12.20)\\
&= \infty + \frac{1}{4} \lim_{x\to 0^+} \frac{1}{x} && (12.21)\\
&= \infty + \frac{1}{4} \cdot \infty && (12.22)\\
&= \infty + \infty && (12.23)\\
&= \infty. && (12.24)
\end{aligned}
$$

The first line is algebra, and the next line is the addition rule. Note that we don't know whether the sum will be well-defined[2] at this stage, but we proceed crossing our fingers. Then I kept simplifying the lefthand term in the summation, knowing that $\lim 1/x^2 = \infty$ and using the scaling law. Line (12.20) follows from the puncture law. Then I use the scaling law, and then my knowledge of $\lim_{x\to 0^+} \frac{1}{x}$. The last few lines are following the arithmetic of ∞.

[2]For example, if at the end we find a sum of the form $\infty - \infty$, we are at a loss—this expression is not defined.

Example 12.1.4. Let's compute

$$\lim_{x\to 3^+} \frac{5x}{x-3}.$$

We have the following string of equalities:

$$\begin{aligned}
\lim_{x\to 3^+} \frac{5x}{x-3} &= \lim_{x\to 3^+} 5x \cdot \lim_{x\to 3^+} \frac{1}{x-3} && (12.25)\\
&= 15 \cdot \lim_{x\to 3^+} \frac{1}{x-3} && (12.26)\\
&= 15 \cdot \infty && (12.27)\\
&= \infty && (12.28)
\end{aligned}$$

The first equality is the product law for limits—note that we did not know[3] that we are allows to use it until the second-to-last line, but we tried computing it anyway (and got lucky that it worked!). (12.26) is evaluating the limit for $5x$, which we knew how to do already. (12.27) is using our knew limit law for functions of the form $1/(x-C)$. Note that C *is* also where we're taking the limit—this is an important part of the law.

The last equality is using the arithmetic rules for ∞.

Example 12.1.5. Let's compute

$$\lim_{x\to 3^-} \frac{5x}{x-3}.$$

We have the following string of equalities:

$$\begin{aligned}
\lim_{x\to 3^-} \frac{5x}{x-3} &= \lim_{x\to 3^-} 5x \cdot \lim_{x\to 3^-} \frac{1}{x-3} && (12.29)\\
&= 15 \cdot \lim_{x\to 3^-} \frac{1}{x-3} && (12.30)\\
&= 15 \cdot -\infty && (12.31)\\
&= -\infty && (12.32)
\end{aligned}$$

[3]We did not know we could use it because we did not know whether the product $\lim_{x\to 3} 5x \cdot \lim_{x\to 3^+} \frac{1}{x-3}$ would yield something non-sensical like $0 \cdot \infty$ upon simplification. When the product *is* sensible, we can safely rely on the product law.

The first equality is the product law for limits—note that we did not know[4] that we are allows to use it until the second-to-last line, but we tried computing it anyway (and got lucky that it worked!). (12.30) is evaluating the limit for $5x$, which we knew how to do already. (12.31) is using our knew limit law for functions of the form $1/(x-C)$. Note that C *is* also where we're taking the limit—this is an important part of the law.

The last equality is using the arithmetic rules for ∞.

[4] We did not know we could use it because we did not know whether the product $\lim_{x\to 3} 5x \cdot \lim_{x\to 3^-} \frac{1}{x-3}$ would yield something non-sensical like $0 \cdot \infty$ upon simplification. When the product *is* sensible, we can safely rely on the product law.

Exercise 12.1.6. Compute—using the limit laws above—the one-sided limits

$$\lim_{x\to 3^-} \frac{\ln x}{x-3} \qquad \text{and} \qquad \lim_{x\to 3^+} \frac{\ln x}{x-3}.$$

Does $\lim_{x\to 3} \frac{\ln x}{x-3}$. exist?

Exercise 12.1.7. Compute

$$\lim_{x\to 0^+} \frac{1}{e^x-1} \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{e^x-1}.$$

Try to think this through *without* using the limit laws above—they won't help.

Does $\lim_{x\to 0} \frac{1}{e^x-1}$ exist?

I want to very carefully walk through this last exercise. How would we compute

$$\lim_{x\to 0^+} \frac{1}{e^x - 1}?$$

Clearly the denominator is causing us trouble. The quotient law doesn't apply because the limit of the denominator is 0.

However, let's think about what's happening to $e^x - 1$ as x approaches 0 from the right. When $x > 0$, we know that $e^x > e^0$. In other words, $e^x > 1$.

Thus, as x approaches 0 from the right, the denominator remains *positive*, but shrinks to zero. (As x approaches 0 from the right, e^x shrinks in size, and e^x becomes closer and closer to 1 while remaining larger than 1. As a result, $e^x - 1$ becomes closer and closer to 0 while remaining positive.)[5]

So $\frac{1}{e^x-1}$, as we shrink x to 0 from the right, is positive, and growing larger and larger (because we are dividing 1 by smaller and smaller numbers). This intuition suggests

$$\lim_{x\to 0^+} \frac{1}{e^x - 1} = \infty.$$

Likewise, as x approaches 0 from the left, e^x is less than 1, but is growing in size to 1. Thus $e^x - 1$ is negative, but approaching 0. So we conclude

$$\lim_{x\to 0^+} \frac{1}{e^x - 1} = -\infty.$$

We will see how to compute this more rigorously next time, when we also talk about limits *at* $x = \pm\infty$.

[5]Make sure you understand this!

12.2 Preparation for Lecture 13: Finding absolute extrema

Remark 12.2.1. What we do in lecture this week will run parallel to (and not intersect, yet) what you will learn in these preparations. In your preparations and quizzes, you will practice another application of derivatives, which is finding absolute extrema.

Here is a problem you might encounter in all kinds of situations, especially in engineering, business, or medicine:

You have a way $f(x)$ to measure outcome, and this outcome depends on an input number x. What value(s) of x implement the best outcome?

Here are some examples:

1. x is the hourly wage of managers at your restaurant, and $f(x)$ is a function measuring output of worker productivity. (This could be measured by combining things like absences, restaurant revenue, tip percentages, time required to complete a food order, or some numerical combination of these.)

2. x is the concentration of Lithium salts like $LiPF_6$ in an electrolyte solution for batteries, and $f(x)$ is a function measuring the efficacy of the resulting battery (this could be measured in voltage, or in longevity, or some combination of various factors).

3. x is the dosage of insulin, and $f(x)$ measures blood sugar levels in the blood stream thirty minute after the insulin intake.

"Best" outcome is entirely subjective, and is in the realm of human sciences more than mathematics. But once you decide what "best" means, you can apply mathematics to solve the problem.

In preparing for Lecture 13 (and in labs this week) you'll learn how to find "best" outcomes when best means "the largest possible," or "the smallest possible" value of $f(x)$. Mathematically, this translates into the problem of finding **absolute extrema** of a function.

Definition 12.2.2. The *absolute maximum* of a function f is the largest value that f takes (if such a value exists).

The *absolute minimum* of a function f is the smallest value that f takes (if such a value exists).

(Confusingly, we will sometimes also call x an absolute minimum/maximum if $f(x)$ is an absolute minimum/maximum. **Be warned** that, of course, x and $f(x)$ are two very different things.)

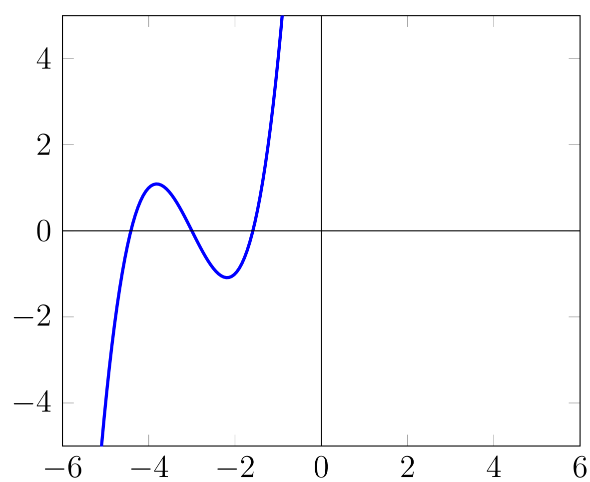
An absolute minimum or maximum of f is called an *extremum*, or *absolute extremum* of f.

(The plural of extremum is "extrema.")

Remark 12.2.3. Often, we will restrict our attention to only a part of the domain of f; most often a closed interval. Then absolute minima/maxima are the smallest/largest values that f takes on that closed interval.

12.2.1 Finding absolute extrema

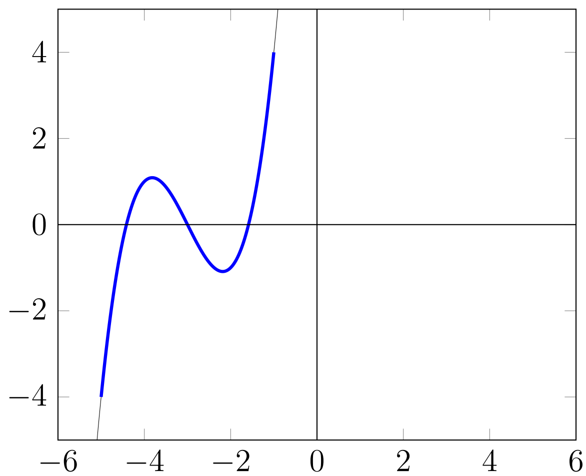
Example 12.2.4. Below is a graph of a function, $f(x)$:



I (arbitrarily) tell you that I am only interested in values of x between -5 and -1, inclusive. (That is, x must be in the interval $[-5, -1]$.)

Problem: For which values of x in $[-5, -1]$ is $f(x)$ largest? For which values is $f(x)$ smallest?

Let's highlight the portion of the function that lies over that interval:



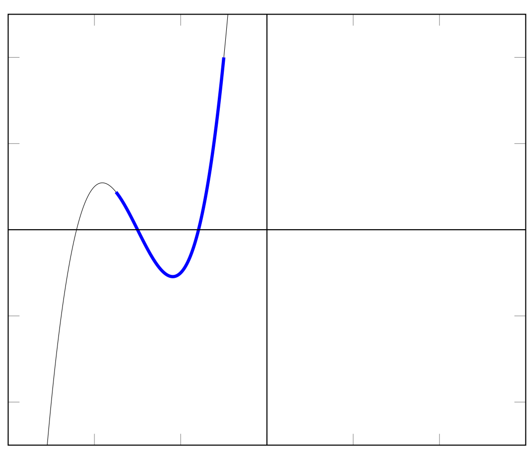
I've drawn thicker the portion of the graph where the x coordinate is in $[-5, -1]$.

As you can tell from the graph, the largest value of $f(x)$ on the interval $[-5, -1]$ is obtained at an endpoint: $x = -1$.

Likewise, the smallest (i.e., most negative) value of $f(x)$ on the interval $[-5, -1]$ is obtained at the other endpoint: $x = -5$.

That was easy. Next, we note that **our answers change based on the interval we choose.**

For example, what if we were only concerned with those x along the interval $[-3.5, -1]$?



As drawn above, the minimum is obtained at the local minimum (the trough). And the maximum is attained at the rightmost endpoint (at $x = -1$), while the other endpoint is neither a maximum nor a minimum.

Thinking about these examples, we realize **how we should look for extrema for differentiable functions on a closed interval**, even if we cannot visualize the graph of the function:

1. Find the critical points of f inside the interval, and compute the values of f at those critical points.

2. Compute the value of f at the endpoints of the interval.

3. Compare these values to conclude which points are absolute extrema.

This is using the following principle: If f is differentiable, **The extrema of f are either critical points or endpoints.**

Example 12.2.5. Let $f(x) = x^3e^x$. Find the absolute extrema of f along the interval $[-10, 1]$.

Let us first find the critical points—i.e., the points where $f'(x) = 0$. We first take the derivative to find[6]

$$f'(x) = (x^3 + 3x^2)e^x.$$

Because e^x is always positive, $f'(x)$ equals zero only when x^3+3x^2 equals zero. Factoring, this happens only when $x^2(x + 3)$ equals zero. Thus, $f'(x) = 0$ when $x = 0$ or $x = -3$.

At this point, it is important to **check whether the critical points are in the relevant interval**. Indeed, both 0 and -3 are in $[-10, 1]$, so we should compute their values. We have that

$$f(0) = 0, \qquad f(-3) = (-3)^3e^{-3} = \frac{-27}{e^3}.$$

(Note that I am not even checking which of these critical points are local minima, maxima, or otherwise; I just care about which values are biggest or smallest, so I go on.)

Now I just need to check what the values of f are at the endpoints of my given interval.

We have

$$f(1) = (1)^3e^1 = 1 \cdot e = e$$

and

$$f(-10) = (-10)^3e^{-10} = \frac{-1000}{e^{10}}.$$

The final step is to compare all the values we found:

(i) $f(0) = 0$,

(ii) $f(-3) = \frac{-27}{e^3}$,

(iii) $f(1) = 3$,

(iv) $f(-1) = \frac{-1000}{e^{10}}$.

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[6] I omitted the work; you are now expect to be able to compute the derivative!

The largest of these values is $f(1) = 3$ (the other values are 0 or negative), so we conclude that the absolute maximum of f along the interval $[-10, 1]$ is attained at $x = 1$.

But which of the four numbers above are smallest? Now we are in the tough situation of comparing two negative numbers with strange expressions:

$$\frac{-27}{e^3} \qquad \text{and} \qquad \frac{-1000}{e^{10}}.$$

Which is more negative? Well, $e > 2$, so $e^{10} > 2^{10}$, while $2^{10} = 1028$. So

$$\frac{1000}{e^{10}} < \frac{1000}{1028} < 1.$$

This means

$$\frac{-1000}{e^{10}} > -1.$$

This gives us some sort of comparison; it may be useful!

For the other value: Let's note $e < 3$, so $e^3 < 3^3$, so $e^3 < 27$. Thus

$$\frac{27}{e^3} > \frac{27}{27} = 1.$$

This means

$$\frac{-27}{e^3} < -1.$$

Thus,

$$\frac{-27}{e^3} < -1 < \frac{-1000}{e^{10}}.$$

So the minimum is obtained at $x = -3$.

(Rest assured: You won't always have to perform delicate inequalities using e; I just wanted to show you the strength of what's possible!)

Example 12.2.6. Find the absolute extrema of $f(x) = x^3 + 3x^2 - 2$ along the interval $[-1, 4]$.

Let's find the critical points. We first find the derivative:

$$f'(x) = 3x^2 + 6x.$$

This factors: $f'(x) = 3x(x + 2)$. So the critical points are at $x = 0$ and $x = -2$ (because those are the x values for which $f'(x)$ equals zero.) Note that $x-2$ is **outside the interval** we are interested in, so we discard it as a possible absolute extremum.

Now we compute the values of f at the critical points, and at the end-points of the interval:[7]

$$f(0) = 0^3 + 3(0^2) - 2 = -2.$$

$$f(-1) = (-1)^3 + 3(-1)^2 - 2 = -1 + 3 - 2 = 0.$$
$$f(4) = (4)^3 + 3(4)^2 - 2 = 64 + 48 - 2 = 110.$$

Out of these values, clearly 110 is the largest, and -2 is the smallest. Thus, we conclude:

The absolute maximum of f (along the interval $[-1, 4]$) occurs at $x = 4$ with $f(4) = 110$.

The absolute minimum of f (along the interval $[-1, 4]$) occurs at $x = 0$ with $f(0) = -2$.

For next lecture, I expect you to be able to compute the absolute extrema for the following functions:

(a) $f(x) = x^3 + 3x^2$ along the interval $[-1, 2]$

(b) $f(x) = e^{x^3-3x}$ along the interval $[-3, \frac{5}{2}]$.

(c) $f(x) = 4x^4 - 3x^3$ along the interval $[-1, 3]$.

[7]By the previous paragraph, we ignore $x = -2$.