Lecture 15

15.1 Related rates

Sometimes, the rate of change of one thing depends on the rate of change of another thing.

Exercise 15.1.1. The area of a crop circle is expanding at a rate of 3 meters squared per minute (i.e., $3\text{m}^2/\text{min}$). If you know what the radius of the crop circle is at a certain time, can you tell me how quickly the radius of this crop circle increasing at that time?

Exercise 15.1.2. The area of a different crop circle at time t is given by

$$A(t) = e^{3t},$$

where t is in minutes and the area is in meters squared. At time t, how quickly is the radius of this crop circle increasing?

In tackling each of these problems, we have to think about how the area depends on the radius. Of course, for circles, we know that Area is equal to π times the radius squared. That is,

$$A = \pi r^2.$$

Now, the area is changing with time, so the radius is changing with time, too. We can write

$$A(t) = \pi r(t)^2.$$

(Both area and radius are now expressed as functions of time.)

So let's take the derivative of both sides, with respect to t:

$$A'(t) = \pi 2r(t) \cdot r'(t).$$

(We are using the chain rule on the righthand side!) Dividing both sides by $2\pi r(t)$, we find:

$$r'(t) = \frac{A'(t)}{2\pi r(t)}.$$

So, for the first exercise, when the circle has radius R, we know that the radius is changing as

$$r'(t) = \frac{A'(t)}{2\pi r(t)} = \frac{3}{2\pi R}.$$

For the second exercise, we see that

$$A'(t) = (e^{3t})' = 3e^{3t}.$$

Moreover, we can find $r(t)$ in terms of $A(t)$:

$$r(t) = \sqrt{A(t)/\pi} = \sqrt{e^{3t}/\pi} = \frac{e^{3t/2}}{\sqrt{\pi}}.$$

So

$$r'(t) = \frac{A'(t)}{2\pi r(t)} \tag{15.1}$$

$$= \frac{3e^{3t}}{\frac{e^{3t/2}}{\sqrt{\pi}}} \tag{15.2}$$

$$= \frac{3e^{3t-(3t)/2}}{\frac{1}{\sqrt{\pi}}} \tag{15.3}$$

$$= 3\sqrt{\pi}e^{3t/2}. \tag{15.4}$$

15.2 Exercises (in implicit differentiation)

Exercise 15.2.1. Here is an equation for a hyperbola:

$$3\,(x+1)^2 - 4\,(y-1)^2 \;=\; 2.$$

Using implicit differentiation, find a formula for the slope of the hyperbola in terms of the x and y coordinates.

Exercise 15.2.2. Consider the shape formed by all points (x, y) satisfying

$$y - \sin(xy) = 0.$$

Using implicit differentiation, find a formula for the slope of the tangent line to this shape at a point (x, y).

Solutions to previous page

Solution to 15.2.1. Taking the derivatives of both sides, we find

$$\frac{d}{dx}(3\,(x+1)^2 - 4\,(y-1)^2) = \frac{d}{dx}(2) = 0. \tag{15.5}$$

We implicitly differentiate the lefthand side:

$$\frac{d}{dx}(3\,(x+1)^2 - 4\,(y-1)^2) = 3 \cdot 2(x+1) - 4 \cdot 2(y-1)y' \tag{15.6}$$
$$= 6(x+1) - 8(y-1)y'. \tag{15.7}$$

From the first equation, we conclude $0 = 6(x+1) - 8(y-1)y'$. By subtracting $6(x+1)$ from both sides and then dividing by $-8(y-1)$, we conclude

$$y' = \frac{6(x+1)}{8(y-1)}.$$

This is our answer. At a point (x, y), the above fraction is the slope of the hyperbola at that point.

Solution to 15.2.2. We have that

$$\frac{d}{dx}(y - \sin(xy)) = y' - (\sin(xy))' \tag{15.8}$$
$$= y' - \cos(xy) \cdot (xy)' \tag{15.9}$$
$$= y' - \cos(xy) \cdot (y + xy'). \tag{15.10}$$

To reach the equality in (15.9), we use the chain rule. The next equality is the product rule. So we conclude

$$0 = y' - \cos(xy) \cdot (y + xy').$$

So

$$0 = y' - y'(x\cos(xy)) - y\cos(xy).$$

Now we can subtract $y\cos(xy)$ from both sides and divide by $(1 - x\cos(xy))$ to find

$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}.$$

15.3 Exercise (in related rates)

Exercise 15.3.1. The surface area of a long rod of length l and radius r is given by

$$A = 2\pi rl$$

(we are ignoring the ends of the rod). The volume of this rod is given by

$$V = \pi r^2 l.$$

(a) Suppose that the length l of a rod is growing at 500 nanometers per hour, and that the radius is growing at 3 nanometers per hour. How fast is the area of the long rod changing? (State your answer in terms of r and l. Your units should be in nanometers-squared per hour.)

(b) Suppose a long rod is growing in length at 500 nanometers per hour, and that the radius is growing at 3 nanometers per hour. How fast is the volume of the long rod changing? (Your units should be in nanometers-cubed per hour.)

(c) The *surface-area-to-volume* ratio is an important ratio for cells. Let's model the axon of a neuron as a long rod of length l and radius r, so that the surface-area-to-volume ratio of the axon is given by

$$A/V.$$

If the neuron is growing in length at 500 nanometers per hour, and the radius is growing at 3 nanometers per hour, how is the surface-area-to-volume ratio changing? (Your units should be in units per nanometer per hour.)

Solutions to previous page

(a) Area, radius, and length are all functions of time, so let's write

$$A(t) = 2\pi r(t)l(t).$$

By using the product rule, we find that

$$A'(t) = 2\pi(r'(t)l(t) + l'(t)r(t)).$$

Thus, if $l'(t) = 500$ and $r'(t) = 3$ as given, we find that

$$A'(t) = 2\pi(3l(t) + 500r(t)).$$

That's our answer. The area is changing at a rate of $2\pi(3l+500r)$ nanometers-squared per hour.

(b) Again we can write $V(t) = \pi(r(t))^2 l(t)$. Taking the derivative, we find that the rate of change of V at time t is given by

$$V'(t) = \pi[2r'(t)r(t)l(t) - r(t)^2 l'(t)].$$

Plugging in $r'(t) = 3$ and $l'(t) = 500$, we find

$$V'(t) = \pi[6r(t)l(t) - 500r(t)^2] = \pi r(t)[6l(t) - 500r(t)].$$

(c) We see that $A(t)/V(t) = 2/r$. So

$$(\frac{A(t)}{V(t)})' = (\frac{2}{r(t)})' = \frac{-2r'(t)}{r(t)^2}$$

where we used the quotient rule at the end. Thus, if $r'(t) = 3$, the rate of change of this ratio is given by

$$\frac{-6}{r(t)^2}.$$

15.4 Preparation for Lecture 16: Mean Value Theorem

For next time, you will learn about the Mean Value Theorem.

15.4.1 Real world thought experiment

Suppose you go on a drive. You drive a total of 70 miles over a 60 minute period. Of course, if you've ever been in a car, you know that your speed is rarely constant. We speed up, we brake. Regardless of *how exactly* you drive during those 60 minutes, we can still ask the following

Question: Were you driving at a speed of 70 miles per hour at *some* point of your drive?

Intuitively, the answer is yes. If you were driving less than 70 miles per hour the whole time, there's no way you could have driven a distance of 70 miles in the span of one hour. (For example, if your max speed were 65 mph, the farthest you could drive in an hour is 65 miles!) Now, maybe you hit 70mph early on, maybe you hit it later in the drive; who knows. But you must have been going that fast at some point.

To summarize: If we know where you began, where you ended, and how long it took you, we *don't* know how exactly you drove, but we *do* know that you reached a certain speed at some point.

15.4.2 Mathematical translation

So let's suppose that f is some function. In the example above, f could be a function that takes the time as an input, and outputs where you are.

What we know is

1. A starting time a,
2. An ending time b,
3. Where you began, $f(a)$, and
4. Where you ended, $f(b)$.

Based on this information, we can conclude: At some point between a and b (inclusive), you had a speed of $(f(b)-f(a))/(b-a)$. In other words, there is some moment c in the interval $[a, b]$ such that $f'(c) = (f(b)-f(a))/(b-a)$.

We'll state this as a theorem:

Theorem 15.4.1 (Mean value theorem)**.** Let f be a function that is differentiable. Then for any two inputs a and b with $a < b$, we can conclude the following:

There is some c in the interval $[a, b]$ so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \tag{15.11}$$

Example 15.4.2. At noon yesterday, the temperature in San Marcos was 60 degrees Farenheit. At noon today, the temperature was 62 degrees Farenheit. Over those 24 hours, the temperature changed by a total of +2 degrees Farenheit. So, by the mean value theorem (assuming that temperature is a differentiable function of time), we know there was *some moment* in those 24 hours at which the temperature was changing at a rate of $\frac{2}{24}$ degrees per hour. That is, a rate of 1/12 degrees per hour.

(That does not mean that the temperature changed 1/12 degrees in some hour; it means at some *moment* in time, 1/12 deg/hr is how fast the temperature was changing.)

In this example, c is that moment, a is noon yesterday, b is noon today, $f(a)$ is 60, $f(b)$ is 62, and $f(x)$ measures the temperature at time x (measured in hours).

Expectation 15.4.3. You are expected to read and to understand the statement of Theorem 15.4.1 above.

15.4.3 Functions that aren't constant

Definition 15.4.4 (Constant functions)**.** We'll call a function *constant* if for every pair of input values, the outputs are the same. That is, f is constant if $f(a) = f(b)$ for every choice of a and b.

A less complicated way to visualize this is that the graph of f is a horizontal line.

But visualizations can be misleading, so I'm going to use the complicated definition. For next time, I want you to be able to prove the following:

Proposition 15.4.5. Suppose f is differentiable. If we know that f is *not* a constant function, then there is some c for which $f'(c)$ is not zero.

Proof. Suppose we know that f is not constant. That means that there are two different inputs that have different outputs. Let's call these inputs a and b, so that the outputs $f(a)$ and $f(b)$ are not equal.

Because f is differentiable, we can use the mean value theorem. The theorem tells us that there is some input c for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

On the right, the numerator is not zero, because we know that $f(b) \neq f(a)$. This means that the fraction on the righthand side is some fraction with a non-zero number in the numerator. Such fractions never equal zero. So $f'(c)$ is not zero.

That finishes the proof! □

For next time, I expect you to be able to write a proof of Proposition 15.4.5 using the mean value theorem.

15.4.4 There are many ways to write a proof for a statement

As with any good medium of expression, there are *many* ways to write a proof. But the *content* must be solid. Below are some examples.

By the way, the white box at the end means "end of proof." Sometimes, people also write "QED" (quod erat demonstrandum) at the end of a proof.

Proof. Because f is not constant, we can find two numbers $a \neq b$ with $f(a) \neq f(b)$. We can note

$$\frac{f(b) - f(a)}{b - a}$$

is a number that does not equal zero, because the numerator is not zero.

On the other hand, the mean value theorem says that (because f is differentiable) there is some number c satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The righthand side—as we just said—is not zero. Thus $f'(c) \neq 0$. Putting everything together, we see that there is some point c at which $f'(c) \neq 0$. □

Proof. By the mean value theorem, for any two points a and b, we know there is some c in $[a, b]$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

So we are finished if we can find a, b so that $f(b) - f(a)$ does not equal zero (for then $f'(c)$ will be non-zero). Well, we know that a and b satisfying $f(b) - f(a)$ exist because $f$is non-constant. QED □

Proof. We know the following:

1. There are two numbers a and b so that $f(b) - f(a) \neq 0$.

2. For some number c between a and b,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3. The fraction $\frac{f(b)-f(a)}{b-a}$ does not equal zero.

The first statement is true because f is not a constant function. The second statement is true by the Mean value theorem (which we can use because f is differentiable). The third statement is true because the fraction has a non-zero numerator (by the first statement).

Putting 2. and 3. together, we have found a number c so that $f'(c)$ does not equal zero. QED. □