Lecture 16

The mean value theorem and logic

In preparation for today, you learned about the mean value theorem:

Theorem 16.0.1 (Mean Value Theorem)**.** If f is differentiable, then for all $a \neq b$, there exists c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

And then you proved for today (and proved on the quiz):

Proposition 16.0.2. If f is not a constant function, then there is some value of x for which $f'(x) \neq 0$.

16.1 Some logic

We're going to talk about logic for a second.

16.1.1 Implications

Here are two statements:

1. If R is a rectangle, then R is a square.
2. If R is a square, then R is a rectangle.

Only one of these statements is true!
Most meaningful logical statements can be written in the form

$$\text{If BLAH, then BLERG}$$

where BLAH and BLERG are statements with some truth value. To be fancy, we often write instead

$$\text{If } P\text{, then } Q$$

where P and Q are fancier stand-ins for BLAH and BLERG. For example, we obtained the above two statements by taking

1. $P =$ "R is a rectangle" and $Q =$ "R is a square."
2. $Q =$ "R is a rectangle" and $P =$ "R is a square."

To be lazy, instead of writing "If P then Q," we may sometimes write

$$P \implies Q.$$

Warning 16.1.1. The following all mean the exact same thing:

- If P, then Q.
- If P is true, then Q is true.
- Given P, we know Q holds.
- P implies Q.

- If we take P as a hypothesis, Q follows as a conclusion.
- In the situation of P, Q is true.
- Suppose P. Then Q is true.
- Whenever P is true, Q is true, too.
- $P \implies Q$.

Remark 16.1.2. Most useful mathematical statements are of the form $P \implies Q$ (or, equivalently, any of the above re-phrasings of this implication). For example,

1. If $f(x) = e^x$, then $f'(x) = e^x$.
2. If a number is even, then it is divisible by 2.

However, as you can see from Warning 16.1.1, the statement $P \implies Q$ may be rephrased in many different ways. This is nice as a matter of poetry and of language; but it can be difficult for a student to pick up on the meanings of statements.

16.1.2 Converses

Given a statement of the form $P \implies Q$, the *converse* is the statement $Q \implies P$.

Example 16.1.3. The converse of "If R is a rectangle, then R is a square" is the statement "If R is a square, then R is a rectangle."

Consider the statement "Every square is a rectangle." The converse statement is "Every rectangle is a square ."

The converse of the converse of a statement is the original statement.

Warning 16.1.4. As you can see from the example, even if a statement is true, the converse statement may be false!

16.1.3 Contrapositives

The *contrapositive* of the statement $P \implies Q$ is the statement

$$\text{not } Q \implies \text{not } P.$$

Warning 16.1.5. "Not Q" is an instance of mathematicians being lazy. For example, if Q is the statement

"R is a rectangle"

then not Q would be the statement

"R is not a rectangle"

In fact, the shorthand "not Q" would be better translated as "Q is not true."

In other words, the contrapositive to $P \implies Q$ is "Q is not true $\implies P$ is not true."

Example 16.1.6.

1. Consider the statement "If R is a square, then R is a rectangle." The contrapositive of this statement is "If R is not a rectangle, then R is not a square."

2. Consider the statement "If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$." The contrapositive of this statement is "If $f'(x)$ does not equal $\cos(x)$, then $f(x)$ does not equal $\sin(x)$."

3. Consider the statement "If $x = 3$, then $x \geq 2$." The contrapositive is "If $x < 2$, then $x \neq 3$."

The greatest fact in logic is: A statement is true whenever its contrapositive is true. A contrapositive is true whenever the original statement is.

Put another way, **a statement is always logically equivalent to its contrapositive.**

Exercise 16.1.7. Find the contrapositives to the following statements:

(a) If f is not constant, then there is some point x at which $f'(x) \neq 0$.

(b) If $f(x)$ is the function e^x, then f always outputs a positive number.

(c) If x is an inflection point of f, then $f''(x) = 0$.

16.2 Application: Functions with the same derivative

Now we are going to prove a fantastic fact.

Proposition 16.2.1. If two functions f and g have the same derivative, then $f - g$ is a constant function.

A "Proposition" is a fact that somebody has proven for you. It is true, and you may use it freely. But I want you to not only know true facts; I want you to know why they are true. So let's see a proof of the proposition:

Proof. We first compute a derivative:

$$(f-g)' = f' - g' \tag{16.1}$$
$$= 0. \tag{16.2}$$

This first equality is the addition rule for derivatives. The next equality uses the fact that f and g have the same derivative—that is, $f' = g'$.

This computation shows us that $f - g$ is a function whose derivative is 0.

Now we apply the contrapositive of our quiz: That is, if a function's derivative is 0, then it is constant! Applying the contrapositive to the function $f - g$, we conclude that $f - g$ is constant. QED. □

Example 16.2.2. Let's find all functions whose derivative is $\cos(x)$.

We know one such function: $f(x) = \sin(x)$. If g is any other function with $g'(x) = \cos(x)$, we know that $f - g$ is constant; in other words,

$$g(x) = \sin(x) + C$$

for some real number C.

Examples of such $g(x)$ would be functions like

$$g(x) = \sin(x) + 10, \qquad g(x) = \sin(x), \qquad g(x) = \sin(x) - e, \qquad \text{etc.}$$

And *any* function with $g'(x) = \cos(x)$ is a function obtained by adding a single number to $\sin(x)$.

16.3 Application: Functions with positive derivatives grow

Here is another intuitive, but very useful fact:

Proposition 16.3.1. Suppose that f is differentiable, and that $f'(x)$ is always positive. Then whenever $b > a$, we have that $f(b) > f(a)$.

In other words, if f has positive derivative, then f is increasing.

Proof. Suppose that $f(b) \leq f(a)$ for some $b > a$. Then

$$\frac{f(b) - f(a)}{b - a} \leq 0.$$

This is because the denominator is positive, while the numerator is 0 or negative.

By the mean value theorem, there would then be some c in $[a, b]$ such that $f'(c) \leq 0$. This violates our hypothesis that f' is always positive.

Thus, it could not be true that $f(b) \leq f(a)$ for some $b > a$. In other words, for all $b > a$, we have that $f(b) > f(a)$. QED. □

Example 16.3.2. Let $f(x) = e^x - 3$. The derivative of f is always positive because e^{anything} is positive. The proposition above tells us that f is always increasing.

16.4 Application: Comparing values by comparing derivatives

Here is another intuitive, but very useful fact:

Proposition 16.4.1. Suppose that f and g are two differentiable functions. Suppose also that a is a number satisfying

1. $f(a) \geq g(a)$, and
2. For all $b > a$, we have that $f'(b) > g'(b)$.

Then $f(b) > g(b)$ for all $b > a$.

In words, the proposition says that if f at least as large as g at a, and if f has bigger derivatives than g from a onward, then f always remains bigger than g.

Proof. By (2), we can conclude that $f'(b) - g'(b) > 0$ for all $g > a$.

Applying Proposition 16.3.1 to the function $f - g$, we can thus conclude that

$$f(b) - g(b) > f(a) - g(a)$$

for all $b > a$. The righthand side is ≥ 0 by (1), so

$$f(b) - g(b) > 0$$

for all $b > a$. In other words,

$$f(b) > g(b) \text{ for all } b > a.$$

QED. □

Example 16.4.2. Let's compare the functions $f(x) = e^x$ and $g(x) = x + 1$. These functions agree at $a = 0$. But $f'(b) = e^b > 1$ whenever $b > 0$. Meanwhile, $g' = 1$. So f' has bigger derivative when $b > 0$. In other words, $f(b)$ will always be larger than $g(b)$ for positive choices of b.

16.5 Preparation for Lecture 17: Curve-sketching

Without using a graphing calculator, let's visualize the function

$$f(x) = \frac{x^2+1}{x^2-2}$$

using tools of calculus!

16.5.1 Asymptotes

First, let's find the vertical and horizontal asymptotes. Remember, you do this by computing

1. The limits at $\pm\infty$ (to find horizontal asymptotes), and

2. The limits where f looks undefined (there are vertical asymptotes if this limit is $\pm\infty$).

(1) You can compute that the horizontal asymptote is 1, and that f approaches 1 near both ∞ and $-\infty$. Here's the computation for the limit at $-\infty$; I'll leave the other limit to you!

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2+1}{x^2-2} &= \lim_{x\to-\infty} \frac{x^2+1}{x^2-2} && (16.3)\\
&= \lim_{x\to-\infty} \frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}} && (16.4)\\
&= \frac{\lim_{x\to-\infty} 1+\frac{1}{x^2}}{\lim_{x\to-\infty} 1-\frac{2}{x^2}} && (16.5)\\
&= \frac{1+0}{1-0} && (16.6)\\
&= 1. && (16.7)
\end{aligned}$$

(You *do* need to compute both limits, because there may be horizontal asymptotes with different heights.)

(2) f potentially has asymptotes where the denominator is zero—that is, when $x = \pm\sqrt{2}$. There are *four* one-sided limits to compute here. I will compute one for you, and tell you the answer for the rest:

$$\lim_{x\to-\sqrt{2}^-}\frac{x^2+1}{x^2-2} = \lim_{x\to-\sqrt{2}^-}\frac{x^2+1}{x^2-2} \tag{16.8}$$

$$= \frac{\lim_{x\to-\sqrt{2}^-} x^2+1}{\lim_{x\to-\sqrt{2}^-} x^2-2} \tag{16.9}$$

$$= \frac{2+1}{0^+} \tag{16.10}$$

$$= \frac{3}{0^+} \tag{16.11}$$

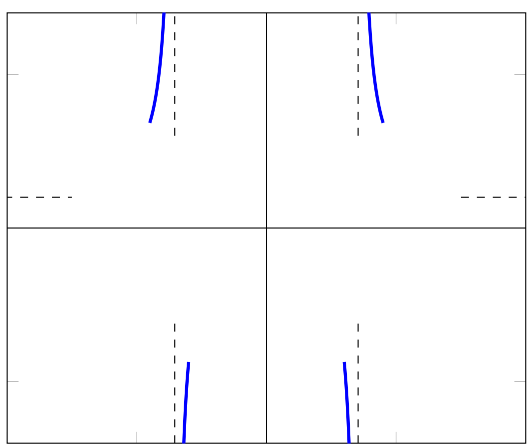
$$= \infty. \tag{16.12}$$

The thing that requires most explanation is probably how we got to line (16.10). As x approaches $-\sqrt{2}$ from the left, x^2 approaches 2 *from the right*; that is, x^2 is shrinking toward 2. Thus, $x^2 - 2$ is approaching 0 from the right. This is why the denominator becomes 0^+.

The three other one-sided limits can be computed to be

$$\lim_{x\to-\sqrt{2}^+}\frac{x^2+1}{x^2-2} = -\infty, \qquad \lim_{x\to\sqrt{2}^-}\frac{x^2+1}{x^2-2} = -\infty, \qquad \lim_{x\to\sqrt{2}^+}\frac{x^2+1}{x^2-2} = \infty.$$

Based purely on these computations, we can begin to visualize the graph of $f(x)$:



Tbe dashed lines show where the asymptotes are; the thick blue lines are the beginnings of the graph. Note that I haven't yet drawn the parts of the graph near horizontal $\pm\infty$; this is because while I know the asymptotes, I don't know how f approaches the asymptotes (for example, f could oscillate close to the asymptote, or approach from above, or from below).

16.5.2 Concavity

Now I'd recommend computing the concavity of the function to get a good feel for shape.

Remember, concavity is dictated by whether the second derivative is positive or negative. So let's compute the second derivative.

First, we compute f':

$$\left(\frac{x^2+1}{x^2-2}\right)' = \frac{(x^2+1)'(x^2-2)-(x^2-2)'(x^2+1)}{(x^2-2)^2} \tag{16.13}$$
$$= \frac{(2x)(x^2-2)-(2x)(x^2+1)}{(x^2-2)^2} \tag{16.14}$$
$$= \frac{(2x)(x^2-2-(x^2+1))}{(x^2-2)^2} \tag{16.15}$$
$$= \frac{(2x)(-3)}{(x^2-2)^2} \tag{16.16}$$
$$= \frac{-6x}{(x^2-2)^2} \tag{16.17}$$

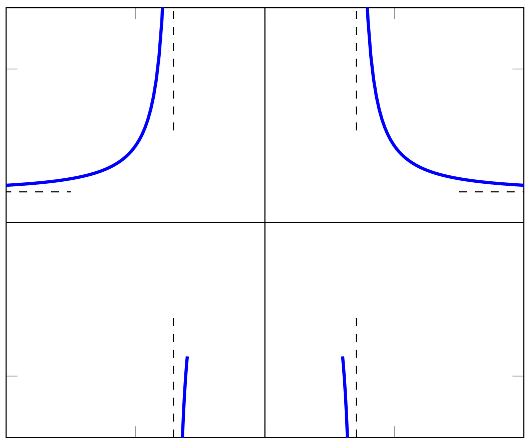
The second derivative is computed as follows:

$$\left(\frac{x^2+1}{x^2-2}\right)'' = \left(\frac{-6x}{(x^2-2)^2}\right)' \tag{16.18}$$
$$= \frac{(-6x)'(x^2-2)^2-(-6x)((x^2-2)^2)'}{(x^2-2)^4} \tag{16.19}$$
$$= \frac{(-6)(x^2-2)^2-(-6x)(x^2-2)(2x)}{(x^2-2)^4} \tag{16.20}$$
$$= \frac{(-6)(x^2-2)[(x^2-2)-(x)(2x)]}{(x^2-2)^4} \tag{16.21}$$
$$= \frac{(-6)(x^2-2)(-x^2-2)}{(x^2-2)^4} \tag{16.22}$$
$$= \frac{6(x^2-2)(x^2+2)}{(x^2-2)^4} \tag{16.23}$$

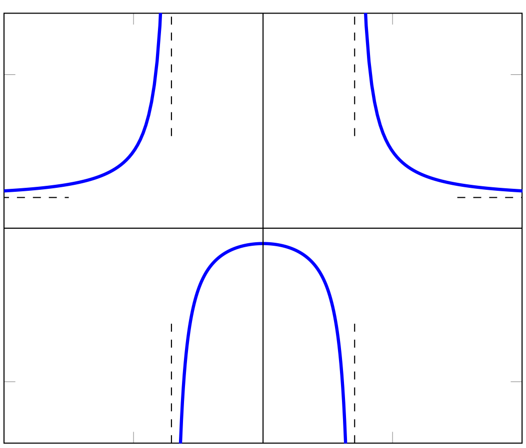
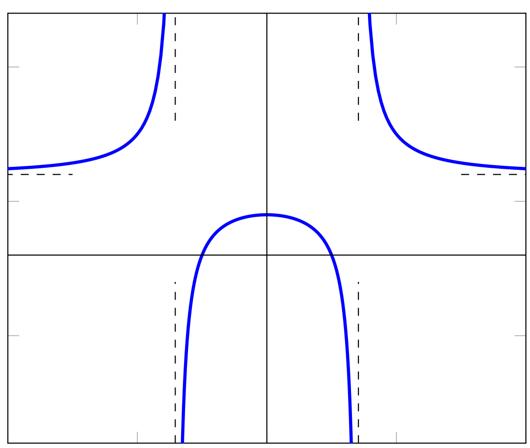
When is this fraction positive, and when is it negative?

The denominator, $(x^2-2)^4$, is always positive, so we can focus on the numerator, $6(x^2-2)(x^2+2)$.

(i) We see that x^2+2 is always positive, while x^2-2 is positive whenever $x^2 > 2$. That is, whenever $|x| > \sqrt{2}$. At this point, we know that the graph of the function must be concave up when $|x| > \sqrt{2}$. So we can begin to draw this portion of the graph:

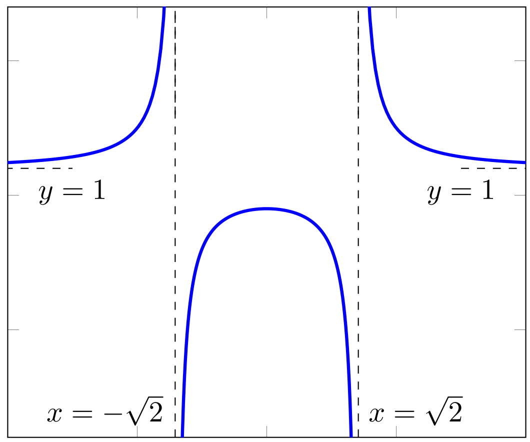


(ii) $6(x^2-2)(x^2+2)$ is negative precisely when $|x| < \sqrt{2}$, by the same reasoning as before; so we know that the function will look concave down in this region. So we can conclude the graph probably looks like one of the following:

At this point, there is still ambiguity in what the graph actually looks like. Where is the local maximum in the middle? At what x-coordinate? And at what y-coordinate?

But, depending on the kind of information you're looking for, you might be satisfied with a vague sketch as follows:



If you want more information, or are asked for more information, you can make a more accurate sketch by finding out things such as:

- Identifying critical points.
- Finding the y- and x-intercepts.
- Labeling inflection points (in our case, we had none).

Let me emphasize one thing: **Your computations by hand are often more reliable than what graphing calculators will show you.** Being able to identify the critical points, the asymptotes, et cetera, can even tell you what frame you should use to look at a graph (e.g., what x values and what y values should your window hold?).

Here is a **summary of curve-sketching**: Identify the asymptotes, identify the concavity of the important regions, and then collect more information if you need (critical points, intercepts, et cetera).

Example 16.5.1. You are told that a function f has the following properties:

(a) $\lim_{x\to-\infty} f(x) = 3$.

(b) $\lim_{x\to\infty} f(x) = 2$.

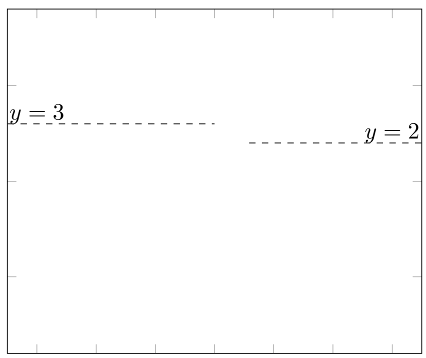
(c) f is continuous and defined everywhere.

(d) $f''(x)$ is positive when x is between -1 and 5

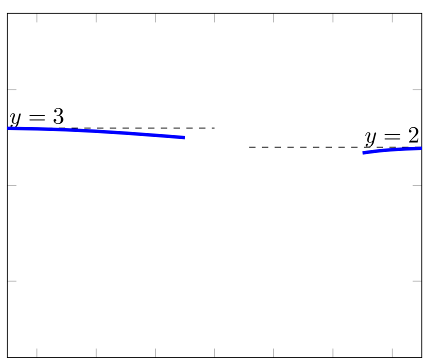
(e) $f''(x)$ is negative when $x < -1$ and when $x > 5$.

Sketch the graph.

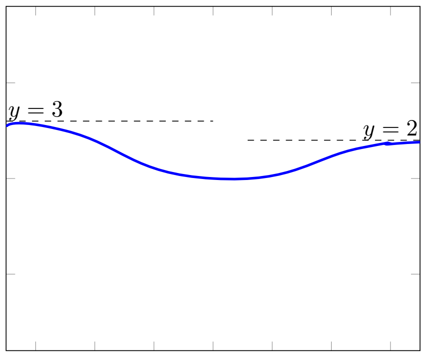
Solution: Based on (a) and (b), we can first draw the horizontal asymptotes, though we don't know how f approaches these asymptotes yet.



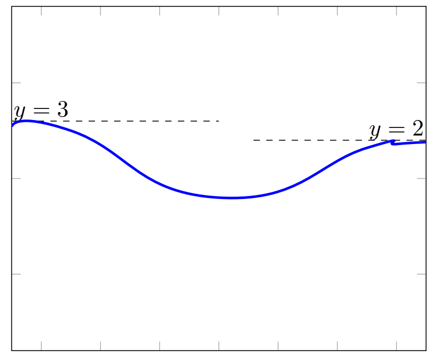
We know there are no vertical asymptotes by (c). By (e), we know that the function looks concave down outside of the interval $[-1, 5]$, so we can begin to draw as follows:



By (d), the rest of the function is concave up. So we sketch a "bowl up" shape:



or



(With the information given, it is impossible to draw the graph of f with complete accuracy, but you see that you get a "feel" for what it looks like!)

For next time, I expect you to able to sketch the following graphs

(a) $f(x) = \frac{1}{x^2-2}$ (explaining *why* the sketch looks the way it does)

(b) $f(x) = \frac{1}{e^x-3}$ (explaining *why* the sketch looks the way it does)

(c) A continuous function f satisfying the following properties:

  (a) $\lim_{x\to\infty} f(x) = 5$
  (b) $\lim_{x\to-\infty} f(x) = -5$
  (c) $\lim_{x\to 2^+} f(x) = \infty$
  (d) $\lim_{x\to 2^-} f(x) = \infty$
  (e) $f''(x) < 0$ when x is less than -10,
  (f) $f''(x) > 0$ when x is between -10 and 2,
  (g) $f''(x) > 0$ when x is larger than 2.

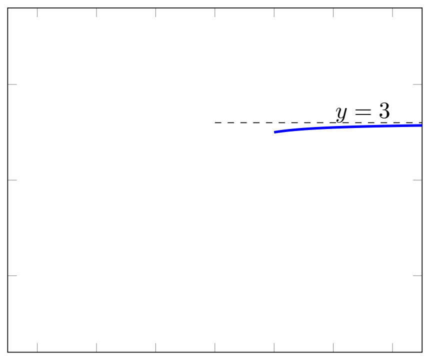
16.6 Appendix: Concavity near ∞

In class, the following question was asked:

If I know

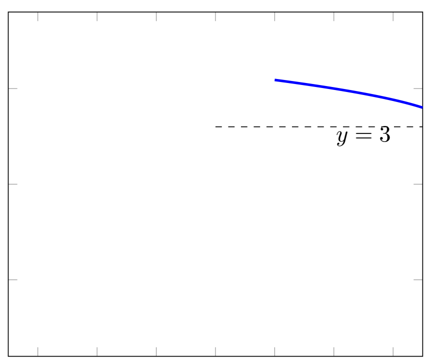
- $\lim_{x\to\infty} f(x) = 3$ and
- $f''(x) < 0$,

why do I know that f has to look like



DRAWING I:

near ∞? For example, why couldn't f look like the following?



DRAWING II:

In this appendix, I claim: **Drawing II can never happen** while f satisfies the two properties above. That is, if f is concave down and has a horizontal asymptote, f must approach that asymptote from *underneath* the asymptote, not from above.

Here is a great place for *proof*. Let's try to put into mathematical language what picture you're drawing in Drawing II: You seem to be drawing a function with

1. $f'' < 0$ for x larger than (for example) 5, and

2. $f' < 0$ for x larger than 5.

I want to emphasize that the role of 5 could be swapped with any number; so let's just call that number a from now on.

I claim the following:

Proposition 16.6.1. Suppose that f is a function such that, for some real number a, we have

1. $f''(x) < 0$ for all $x > a$, and

2. $f'(x) < 0$ for all $x > a$.

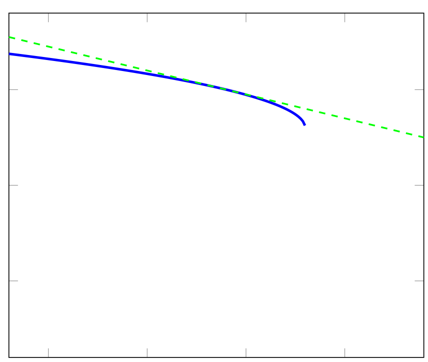
Then $\lim_{x\to\infty} f(x) = -\infty$. In particular, f does not have a horizontal asymptote as $x \to \infty$.

Proof. Let b be any number bigger than a, and let's let $M = f'(b)$. (That is, M is the slope of the tangent line to f at b.)

Then we can define another function called

$$g(x) = M(x - b) + f(b).$$

The graph of g is a line—a line with slope M, and which passes through the point $(b, f(b))$.



Above, the graph of g has been represented as a dashed green line, and the graph of f as a solid blue curve. Note g has negative slope because of hypothesis 2. of the Proposition.

Now, because $f''(x) < 0$, we know that $f'(x)$ will be less than $f'(b)$ for all $x > b$. This is a consequence of one of the statements we saw last lecture (as an application of the mean value theorem)! It's because you can think of $h = f'(x)$ as a function; and when $h'(x) < 0$ for all $x > a$ (because of Hypothesis 1), we know that h is decreasing in value for all $x > a$, so $h(x)$ will be less than $h(b)$ whenever $x > b$.

Now, because $f'(x) < f'(b) = g'(x)$ for all $x > b$, we see that $f(x) < g(x)$ for all $x > b$. (This is using another consequence of the mean value theorem from last lecture.)

So if $f(x) < g(x)$ for all $x > b$, we conclude that

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x).$$

But $g(x) = M \cdot (x - b) + f(b)$. So we see that

$$\begin{aligned}
\lim_{x\to\infty} g(x) &= M \cdot (\lim_{x\to\infty} -b) + f(b) && (16.24)\\
&= M \cdot (\infty - b) + f(b) && (16.25)\\
&= M \cdot (\infty) + f(b) && (16.26)\\
&= -\infty + f(b) && (16.27)\\
&= -\infty. && (16.28)
\end{aligned}$$

(Note that $M \cdot \infty = \infty$ because $M < 0$.)

Putting everything together, we see

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x) = -\infty$$

so $\lim_{x\to\infty} f(x) = -\infty$. □