Lecture 17

L'Hôpital's rule

Exercise 17.0.1. Remember that

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1.$$

For no good reason, let's take the derivative of the top and bottom functions, and then take the limit:

$$\lim_{x \to 0^+} \frac{(\sin(x))'}{(x)'}.$$

What answer do you get?

Exercise 17.0.2. Compute the limit

$$\lim_{x \to \infty} \frac{2x+3}{5x-7}.$$

Let's try taking the derivative of the top and bottom function first, and then take the limit. That is, compute

$$\lim_{x \to \infty} \frac{(2x+3)'}{(5x-7)'}.$$

How do your answers compare?

Exercise 17.0.3. Compute the limits

$$\lim_{x \to \infty} \frac{x^2}{1/x} \qquad \text{and} \qquad \lim_{x \to \infty} \frac{(x^2)'}{(1/x)'}.$$

How do your answers compare?

The first two exercises were promising, but the last one showed that this trick doesn't always work. Here is a theorem that you may use freely; we won't prove it in this class:

Theorem 17.0.4 (L'Hôpital's Rule)**.** Let f and g be functions. If

1. $\lim f(x) = \infty$ and $\lim g(x) = \infty$, or if
2. $\lim f(x) = 0$ and $\lim g(x) = 0$,

then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

In words, if the limit of the denominator and numerator *both* equal ∞, or if both equal zero, then the limit of the fraction may be computed by first taking the derivatives of f and g.

Warning 17.0.5. The hypothesis of L'Hôpital's Rule is important! (The limits of the denominator and numerator must both agree.) You saw in Exercise 17.0.3 an example where the numerator and denominator had different limits; as a result, the limit of the fraction after taking the derviatives was *different* from the limit of the fraction.

Remark 17.0.6. The limits in the statement of L'Hôpital's Rule have no subscripts. This is because I am being lazy. To be explicit: If all the limits are taken at the same point, then the theorem holds.

For example, if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^+} g(x)$ both equal zero, you can apply L'Hôpital's Rule:

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

This works for one-sided limits from the left, and for limits at $\pm\infty$.

Remark 17.0.7. As you may have guessed, "L'Hôpital" is a French name. It is pronounced (roughly) "Lo-pee-tahl." You may not be used to the ô; that is, to the little "hat" on top of the *o*. This symbol is called a *circumflex,* and in French, it is often used when a word *used* to have an *s* right after the circumflex. So for example, in the past, the word "L'Hôpital" would have been spelled "L'Hospital." Yes, that's right; this person's name literally translates to "The Hospital."

Exercise 17.0.8. Evaluate the following limits. Some may involve L'Hôpital's rule; other may not. When you use L'Hôpital's rule, say why you know you can use it (based on the hypotheses of the theorem above).

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

(c) $\lim_{x\to 1^+} \frac{x}{x^2-1}$

(d) $\lim_{x\to-\infty} \frac{1}{2x+3}$

(e) $\lim_{x\to 0^+} x\ln x$

(f) $\lim_{x\to 0^-} \frac{x}{\sin(x)}$

(g) $\lim_{x\to\infty} xe^x$

(h) $\lim_{x\to-\infty} xe^x$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

Solutions

(a) $\lim_{x\to(\pi/2)^+}\frac{(x-\pi/2)\sin(x)}{\cos(x)}$

Evaluating the limit in the numerator and denominator yields

$$\frac{\lim_{x\to(\pi/2)^+}(x-\pi/2)\sin(x)}{\lim_{x\to(\pi/2)^+}\cos(x)} = \frac{(\pi/2-\pi/2)\cdot 1}{0} \tag{17.1}$$

This is 0/0, so we can use L'Hôpital's rule.

$$\begin{aligned}
\lim_{x\to(\pi/2)^+}\frac{(x-\pi/2)\sin(x)}{\cos(x)} &= \lim_{x\to(\pi/2)^+}\frac{((x-\pi/2)\sin(x))'}{(\cos(x))'} && (17.2)\\
&= \lim_{x\to(\pi/2)^+}\frac{\sin(x)+(x-\pi/2)\cos(x)}{-\sin(x)} && (17.3)\\
&= \lim_{x\to(\pi/2)^+}\frac{1+0}{-1} && (17.4)\\
&= -1. && (17.5)
\end{aligned}$$

(b) $\lim_{x\to\infty}\frac{x}{x^2-1}$

Evaluating limits in the numerator and denominator, we obtain ∞/∞, so we can use L'Hôpital's rule.

$$\begin{aligned}
\lim_{x\to\infty}\frac{x}{x^2-1} &= \lim_{x\to\infty}\frac{(x)'}{(x^2-1)'} && (17.6)\\
&= \lim_{x\to\infty}\frac{1}{2x} && (17.7)\\
&= 0. && (17.8)
\end{aligned}$$

You also could have solved the original limit without L'Hôpital's Rule: Just divide top and bottom by x.

(c) $\lim_{x\to 1^+}\frac{x}{x^2-1}$

We *cannot* use L'Hôpital's Rule here because, when evaluating the limits of the numerator and denominator, we arrive at 1/0. This is not 0/0 nor ∞/∞.

But we can still divide top and bottom by x. Then

$$\lim_{x\to 1^+} \frac{x}{x^2-1} = \lim_{x\to 1^+} \frac{x}{x^2-1} \cdot \frac{1/x}{1/x} \tag{17.9}$$
$$= \lim_{x\to 1^+} \frac{1}{x-\frac{1}{x}} \tag{17.10}$$
$$= \frac{\lim_{x\to 1^+} 1}{\lim_{x\to 1^+} x-\frac{1}{x}} \tag{17.11}$$
$$= \frac{1}{\lim_{x\to 1^+} x-\frac{1}{x}}. \tag{17.12}$$

When $x > 1$, we know that $x - 1/x$ is positive. So the denominator approaches 0 from the right.

$$= \frac{1}{0^+} = \infty.$$

Here is another way you could have computed this limit. Note that $(x^2 - 1) = (x + 1)(x - 1)$, and we know that $(x - 1)$ is the factor that is causing the denominator to become 0 in the limit. So let's rewrite things in a way we can try to factor out an $(x - 1)$ from the *numerator*, too:

$$\lim_{x\to 1^+} \frac{x}{x^2-1} = \lim_{x\to 1^+} \frac{x}{(x-1)(x+1)} \tag{17.13}$$
$$= \lim_{x\to 1^+} \frac{x-1+1}{(x-1)(x+1)} \tag{17.14}$$
$$= \lim_{x\to 1^+} \frac{x-1}{(x-1)(x+1)} + \lim_{x\to 1^+} \frac{1}{(x-1)(x+1)} \tag{17.15}$$
$$= \lim_{x\to 1^+} \frac{1}{x+1} + \lim_{x\to 1^+} \frac{1}{(x-1)(x+1)} \tag{17.16}$$
$$= \lim_{x\to 1^+} \frac{1}{1} + \lim_{x\to 1^+} \frac{1}{x^2-1} \tag{17.17}$$
$$= 1 + \lim_{x\to 1^+} \frac{1}{x^2-1}. \tag{17.18}$$

Now note that $x^2 - 1$ approaches 0 from the right when $x \to 1^+$, because if $x > 1$, then $x^2 > 1$. So this limit becomes

$$= 1 + \frac{1}{0^+} = 1 + \infty = \infty$$

just as before.

(d) $\lim_{x\to-\infty}\frac{1}{2x+3}$ We don't need L'Hôpital's Rule: We see this limit is $1/-\infty = 0$. Note that we couldn't have used L'Hôpital's Rule anyway.

(e) $\lim_{x\to 0^+} x\ln x$

Let's rewrite this limit:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \tag{17.19}$$

$$= -\lim_{x\to 0^+}\frac{-\ln x}{1/x}. \tag{17.20}$$

You don't really need to insert the minus sign, but I did so to see that $\lim_{x\to 0^+} -\ln x = \infty$ and $\lim_{x\to 0^+} = \infty$; this shows we can apply L'Hôpital's Rule. Applying said rule, we find

$$-\lim_{x\to 0^+}\frac{-\ln x}{1/x} = -\lim_{x\to 0^+}\frac{(-\ln x)'}{(1/x)'} \tag{17.21}$$

$$= -\lim_{x\to 0^+}\frac{-1/x}{-1/x^2} \tag{17.22}$$

$$= -\lim_{x\to 0^+}\frac{1/x}{1/x^2} \tag{17.23}$$

$$= -\lim_{x\to 0^+}\frac{1/x}{1/x^2}\cdot\frac{x^2}{x^2} \tag{17.24}$$

$$= -\lim_{x\to 0^+}\frac{x}{1} \tag{17.25}$$

$$= -\frac{0}{1} \tag{17.26}$$

$$= 0. \tag{17.27}$$

(f) $\lim_{x\to 0^-}\frac{x}{\sin(x)}$

We already know that $\lim_{x\to 0}\frac{\sin x}{x} = 1$, so we can use that to say

$$\lim_{x\to 0^-}\frac{x}{\sin(x)} = \lim_{x\to 0^-}\frac{1}{\frac{x}{\sin(x)}} \tag{17.28}$$

$$= \frac{1}{\lim_{x\to 0^-}\frac{x}{\sin(x)}} \tag{17.29}$$

$$= \frac{1}{1} \tag{17.30}$$

$$= 1. \tag{17.31}$$

You could also use L'Hôpital's Rule to arrive at

$$\lim_{x\to 0^+} \frac{1}{\cos(x)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

(g) $\lim_{x\to\infty} xe^x$ You don't need L'Hôpital's Rule here; we plainly see that the limit is given by $\infty \cdot \infty = \infty$.

(h) $\lim_{x\to-\infty} xe^x$ This gets trickier, because we find (taking naive limits)

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} x \lim_{x\to-\infty} e^x = (-\infty)\cdot(0)$$

which is undefined. So let's rewrite this limit as a fraction:

$$\lim_{x\to-\infty} xe^x = = \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{17.32}$$
$$= \frac{\lim_{x\to-\infty} x}{\lim_{x\to-\infty} e^x} \tag{17.33}$$
$$= -\frac{\lim_{x\to-\infty} -x}{\lim_{x\to-\infty} e^x} \tag{17.34}$$
$$= -\frac{\infty}{\infty} \tag{17.35}$$

which means we can use L'Hôpital's Rule on this limit. We find:

$$\lim_{x\to-\infty} xe^x = = \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{17.36}$$
$$= \frac{\lim_{x\to-\infty}(x)'}{\lim_{x\to-\infty}(e^x)'} \tag{17.37}$$
$$= \frac{\lim_{x\to-\infty} 1}{\lim_{x\to-\infty} e^x} \tag{17.38}$$
$$= \frac{1}{\lim_{x\to-\infty} e^x} \tag{17.39}$$
$$= \frac{1}{0^+} \tag{17.40}$$
$$= \infty. \tag{17.41}$$
$$\tag{17.42}$$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

Evaluating the numerator and denominator limits, we obtain ∞/∞, so we can use L'Hôpital's Rule. Then we end up with

$$\lim_{x\to\infty} \frac{\ln 5 5^x}{2x}.$$

Taking limits of top and bottom, again we find ∞/∞. So we can use L'Hôpital's Rule *again*. Then we find

$$\lim_{x\to\infty} \frac{(\ln 5)^2 5^x}{2}.$$

The limit of this expression is clearly ∞.

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

This problem is the same work as above, but you use L'Hôpital's Rule three times.

17.1 Preparation for Lecture 18 (Review problems)

For the next quiz, you should be able to answer all of the following true-false questions.

Warning. On the quiz, you will receive +2 points for a correct response, and receive *zero* points for an incorrect response. You may write "I don't know" for +1 point. Remember that true/false questions can be tricky. Also, remember that in math and logic, "True" is the same thing as ALWAYS TRUE, and "False" is the same thing as "NOT always true," put another way, "False" means "this is false in at least one case."

State whether each of the following is true or false:

(a) If "$P \implies Q$" is a true statement, then so is " not $Q \implies$ not P".

(b) If " not $Q \implies$ not P" is a true statement, so is "$P \implies Q$."

(c) If " not $Q \implies$ not P" is a true statement, so is "$Q \implies P$."

(d) The statement "If S is a circle, then S is an ellipse" has converse "If S is an ellipse, then S is a circle."

(e) Fix a function f, an input number a, and another number L. Suppose that **for every** $\epsilon > 0$**, there is a** $\delta > 0$ so that for all $x \neq a$, we have that $|x - a| < \delta \implies |f(x) - L| < \epsilon$. Then $\lim_{a \to x} f(x) = L$.

(f) Fix a function f, an input number a, and another number L. Suppose that **for every** $\delta > 0$**, there is an** $\epsilon > 0$ so that for all $x \neq a$, we have that $|x - a| < \delta \implies |f(x) - L| < \epsilon$. Then $\lim_{a \to x} f(x) = L$.

(g) The function $f(x) = |x|$ is differentiable at $x = 0$.

(h) The function $f(x) = |x|$ is differentiable at $x = 1$.

(i) The function $f(x) = |x|$ is differentiable at $x = 5$.

(j) The function $f(x) = |x|$ is differentiable at $x = -2$.

(k) If $f'(a) = 0$ and $f''(a) < 0$, then a is a local maximum of f.

(l) If $f'(a) = 0$ and $f''(a) = 0$, then a is a local maximum of f.

(m) If $f'(a) > 0$ and $f''(a) = 0$, then a is a local maximum of f.

(n) If $f'(a) > 0$ and $f''(a) < 0$, then a is a local maximum of f.

(o) The derivative of $\cos(x)$ is $\sin(x)$.

(p) For any positive number a, the derivative of $f(x) = a^x$ is given by $f(x)$ again.