Lecture 23

Integration by parts

We saw last time that "u substitution" is nothing more than using the chain rule backwards. However, as you saw in lab, it was somehow algebraically valuable to be able to write things like

$$du = \frac{du}{dx}\,dx$$

when substituting $u = h(x)$.

Today, we will learn about "integration by parts." This is nothing more than the product rule, applied backward. But it will be quite useful.

Recall that the product rule says that if we have two functions $u(x)$ and $v(x)$, then

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Thus, taking the integral of both sides, we find

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

The utility of this is to write

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Another way people write this is as follows:

$$\int u\,dv = uv - \int v\,du.$$

As with u substitution, you can make concrete sense of the above expression by setting $dv = \frac{dv}{dx}\,dx$ and setting $du = \frac{du}{dx}\,dx$.

Example 23.0.1. Let's see how to evaluate $\int x\sin(x)\,dx$. Note that this is not an integral we'd know how to do by u substitution—x is not the derivative of any other function in sight; nor is $\sin(x)$.

However, let's try setting $u(x) = x$ and $v'(x) = \sin(x)$. then we have that

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Knowing that the integral of $\sin(x)$ is $-\cos(x)$, we conclude that $v(x) = -\cos(x)$, so we conclude

$$\int x\sin(x)dx = x(-\cos(x)) - \int 1\cdot(-\cos(x))\,dx \tag{23.1}$$

$$= x(-\cos(x)) + \int \cos(x)\,dx \tag{23.2}$$

$$= -x\cos(x) + \sin(x) + C. \tag{23.3}$$

Indeed, by taking the derivative of (23.3), we can check our work:

$$(-x\cos(x) + \sin(x))' = (-x\cos(x))' + (\sin(x))' \tag{23.4}$$

$$= -x(-\sin(x)) + -1\cos(x) + \cos(x) \tag{23.5}$$

$$= x\sin x. \tag{23.6}$$

Remark 23.0.2. You know you might utilize integration by parts when your integral is the product of two functions. However, which function do you take to be u, and which do you take to be v'?

As a general rule, you should always take

- v' to be a function you know how to integrate, and which becomes *no less* complicated when you take its derivative, and
- u to be a function which seems to become "easier" when you do take the derivative.

For example, things like x^3 tend to become "simpler" as you take derivatives. The successive derivative are $3x^2$, $6x$, 6, and then 0. So these are great candidates for u.

On the other hand, things like $\sin(x)$ don't tend to become easier as you take derivatives—we just cycle through cos and sin with various signs. But sin is a function we certainly know how to integrate, and it becomes no "more" complicated when we do take integrals. Likewise, e^x is a function that certainly does not become simpler as one takes derivatives, but its integral is no more complicated than the original.

A function like $\ln x$ certainly becomes simpler when we take a derivative—$1/x$ is a nice, concrete function. We have not yet seen how to take the integral of $\ln x$, so we probably wouldn't want to make it v'.

Remark 23.0.3. Let's stare again at the formula

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

The power of this formula is that we can exchange an integral with v' in it to an integral with u' in it. This is highly useful if $u'(x)$ is simpler than $u(x)$, and if $v'(x)$ is straightforward to integrate. The point is that if u' is a "easier" function to deal with than u', and if v is no more complicated than v', then you have a better hope of integratin $u'v$ than you do of integrating uv'.

Example 23.0.4. Integrate $\int xe^x\,dx$.

As mentioned in the above remark, let's take $u = x$ (because the derivative of x is simpler than x itself) and $v' = e^x$. Then

$$\begin{aligned} \int xe^x,\ dx &= xe^x - \int e^x dx && (23.7) \\ &= xe^x - e^x + C. && (23.8) \end{aligned}$$

You can check your answer (by taking derivatives) that this is indeed the integral of xe^x.

Exercise 23.0.5. Find the antiderivatives of the following functions.

(a) $\ln x$. (Hint: Take $u = 1$.)

(b) $x\sin(x)$.

(c) $x^2\sin(x)$. (Hint: You may have to do integration by parts twice.)

(d) $\sin(x)e^x$. (Hint: You may have to do integration by parts twice.)

As you can see from the examples above, you often have to do integration by parts multiple times to get your answer.