Lecture 24

Covid-19 updates, and integration by parts, continued

24.1 Welcome back

We're in a new world! I have tried to send a minimal number of emails to reduce the clutter in your inboxes. Everything you need to know is contained in an email with the subject line "[Math 2471] Our Class Going Forward. Office Hours, et cetera."

Let me give the highlights:

(a) All lectures are now via Zoom. The Meeting ID and Zoom Link are in the e-mail.

(b) Let me know immediately if internet access, sufficient bandwidth, or appropriate physical space for engaging in an online lecture, are not available. I will do my best to work with you to find an accommodation that works.

(c) I have changed Monday office hours. Now, my office hours are Mondays 5 PM - 6 PM, and Wednesdays 10 AM - 11 AM. Office hours will be conducted via Zoom, and the Zoom link is on the e-mail I sent out.

(d) Quizzes will now take place at the end of class.

24.2 Integration by parts

Let's go back to "integration by parts." This is nothing more than the product rule, applied backward. But it will be quite useful.

Recall that the product rule says that if we have two functions $u(x)$ and $v(x)$, then

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Thus, taking the integral of both sides, we find

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

(Remember that the indefinite integral is an antiderivative; this is why $\int (u(x)v(x))'dx = u(x)v(x)$.)

The utility of this formula, as it turns out, emerges by subtracting a term from both sides to obtain

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx. \tag{24.1}$$

Equation (24.1) is the most important fact you'll use.

If we solve an integral by using equation (24.1), we say that we are **integrating by parts**. (This is an old-fashioned terminology, but still in use—the "parts" are the u and the v, presumably.) As you can see, integration by parts is useful for integrating products of functions. (For example, (24.1) is telling us that we can integrate the product uv' if we know how to integrate another product, $u'v$.) u substitution, similarly, allowed us to integrate products of functions—but the kinds of products we saw in u substitution were quite special, as one of the factors needed to be a composition $f \circ g$, and the derivative of g needed to be the other factor. Integration by parts is successful in a different, but much wider, array of situations.

Remark 24.2.1. Another way people write this is as follows:

$$\int u\,dv = uv - \int v\,du.$$

As with u substitution, you can make concrete sense of the above expression by setting $dv = \frac{dv}{dx}\,dx$ and setting $du = \frac{du}{dx}\,dx$.

Example 24.2.2. Let's see how to evaluate $\int x\sin(x)\,dx$. Note that this is not an integral we'd know how to do by u substitution—x is not the derivative of any other function in sight; nor is $\sin(x)$.

However, let's try setting $u(x) = x$ and $v'(x) = \sin(x)$. then we have that

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Knowing that the integral of $\sin(x)$ is $-\cos(x)$, we conclude that $v(x) = -\cos(x)$, so we conclude

$$\int x\sin(x)dx = x(-\cos(x)) - \int 1\cdot(-\cos(x))\,dx \tag{24.2}$$
$$= x(-\cos(x)) + \int \cos(x)\,dx \tag{24.3}$$
$$= -x\cos(x) + \sin(x) + C. \tag{24.4}$$

Indeed, by taking the derivative of (24.4), we can check our work:

$$(-x\cos(x) + \sin(x))' = (-x\cos(x))' + (\sin(x))' \tag{24.5}$$
$$= -x(-\sin(x)) + -1\cos(x) + \cos(x) \tag{24.6}$$
$$= x\sin x. \tag{24.7}$$

Remark 24.2.3. You know you might utilize integration by parts when your integral is the product of two functions. However, which function do you take to be u, and which do you take to be v'?

As a general rule, you should always take

- v' to be a function you know how to integrate, and which becomes *no less* complicated when you take its derivative, and
- u to be a function which seems to become "easier" when you do take the derivative.

For example, things like x^3 tend to become "simpler" as you take derivatives. The successive derivative are $3x^2$, $6x$, 6, and then 0. So these are great candidates for u.

A function like $\ln x$ also becomes simpler when we take a derivative—$1/x$ is a nice, concrete function. We have not yet seen how to take the integral of $\ln x$, so we probably wouldn't want to make it v'.

On the other hand, things like $\sin(x)$ don't tend to become easier as you take derivatives—we just cycle through cos and sin with various signs. But sin is a function we certainly know how to integrate, and it becomes no "more" complicated when we do take integrals. Likewise, e^x is a function that certainly does not become simpler as one takes derivatives, but its integral is no more complicated than the original. Thus, if my integrand has a factor of $\sin(x)$, of $\cos(x)$, or of e^x, these are *prime* candidates for v'. (Pun intended.)

Remark 24.2.4. Let's stare again at the formula

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

The power of this formula is that we can exchange an integral with v' in it to an integral with u' in it. This is highly useful if $u'(x)$ is simpler than $u(x)$, and if $v'(x)$ is straightforward to integrate. The point is that if u' is a "easier" function to deal with than u', and if v is no more complicated than v', then you have a better hope of integrating $u'v$ than you do of integrating uv'.

Example 24.2.5. Integrate $\int xe^x\,dx$.

As mentioned in the above remark, let's take $u = x$ (because the derivative of x is simpler than x itself) and $v' = e^x$. Then

$$\int xe^x\,dx = xe^x - \int e^x dx \tag{24.8}$$

$$= xe^x - e^x + C. \tag{24.9}$$

You can check your answer (by taking derivatives) that this is indeed the integral of xe^x.

Exercise 24.2.6. Find the antiderivatives of the following functions.

(a) $\ln x$. (Hint: Take $u = 1$.)

(b) $x\cos(x)$.

(c) $x^2\sin(x)$. (Hint: You may have to do integration by parts twice.)

(d) $\sin(x)e^x$. (Hint: You may have to do integration by parts twice.)

As you can see from the examples above, you often have to do integration by parts multiple times to get your answer.

24.3 Preparation for next time

So far, we have taken integrals of functions over intervals of the form $[a, b]$. For next time, I want you to learn how to take integrals over intervals of the form $[a, \infty)$. (That's right—these are intervals of infinite length!) We'll see how to take integrals over $(-\infty, \infty)$ and $(-\infty, b]$ in class.

Definition 24.3.1. We define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Let's be careful about this. The definition means that finding the integral of f over $[a, \infty)$ is a two-step process:

1. First, you find an expression for $\int_a^b f(x)\,dx$ in terms of b.

2. Then, you take the limit of this expression as b goes to infinity.

Example 24.3.2. Let $f(x) = 1/x^2$ and $a = 1$. Let's evaluate

$$\int_1^\infty \frac{1}{x^2}\,dx.$$

Following the advice above, we first compute the integral $\int_1^b \frac{1}{x^2}\,dx$. (Remember, we have seen the idea that you can think of the integral as a function in b; we are writing down this function here.) We have

$$\begin{aligned}
\int_1^b \frac{1}{x^2}dx &= -\frac{1}{x}\Big|_1^b && (24.10)\\
&= -\frac{1}{b} - \left(-\frac{1}{1}\right) && (24.11)\\
&= -\frac{1}{b} + 1 && (24.12)\\
&= 1 - \frac{1}{b}. && (24.13)
\end{aligned}$$

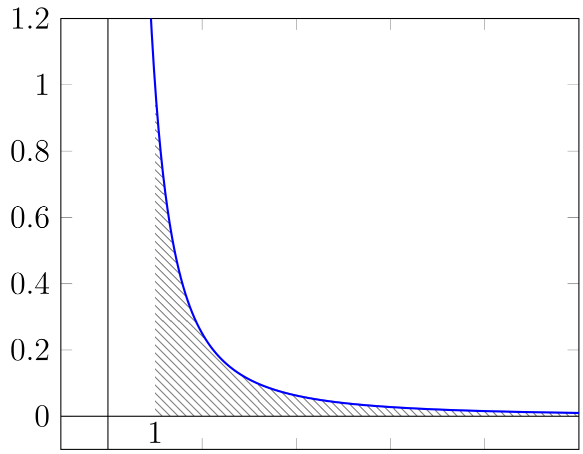
In short, we have

$$\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}.$$

Now we compute the limit:

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) \tag{24.14}$$
$$= 1 - \lim_{b\to\infty}\frac{1}{b} \tag{24.15}$$
$$= 1 - 0 \tag{24.16}$$
$$= 1. \tag{24.17}$$

Let's take a moment to take this in. Remember, integrals are supposed to represent area. And the area represented by $\int_1^\infty \frac{1}{x^2}\,dx$ is drawn as follows:

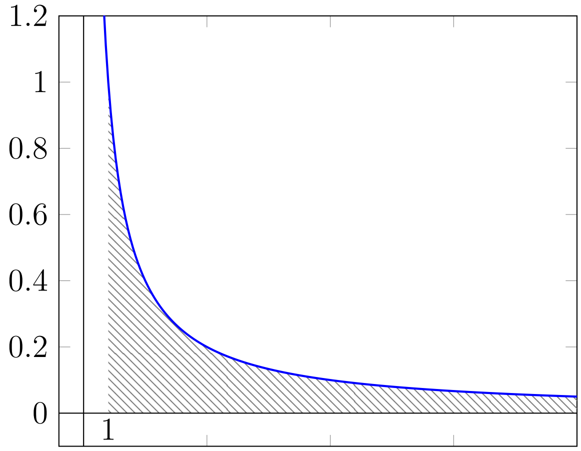


You can think of the shaded region as a fence that goes on forever (the fence begins at $x = 1$, and goes all the way toward ∞), but whose height shrinks to zero as we move to the right. You can interpret our result as saying that even though the fence is infinitely long, the total amount of wood required to build this fence (as measured by area, for example) is finite!

Example 24.3.3. Now let's try to evaluate the integral

$$\int_1^\infty \frac{1}{x}\,dx.$$

Here is a picture of the area represented by this integral. (We are finding the area of the shaded region.)



Just as before, you can think of this shaded region as a fence (or a wall) that is infinitely long, but whose height is shrinking to zero as we move infinitely far to the right. But can you build this fence/wall using a finite amount of material? Let's see.

First, we compute the integral from 1 to b:

$$\int_1^b \frac{1}{x}\,dx = \ln x \Big|_1^b \tag{24.18}$$
$$= \ln b - ln 1. \tag{24.19}$$
$$= \ln b - 0 \tag{24.20}$$
$$= \ln b. \tag{24.21}$$

Thus we find that

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \tag{24.22}$$
$$= \lim_{b\to\infty} \ln b. \tag{24.23}$$

This limit is ∞. (For example, if $b = e^y$, as b goes to infinity, y must also approach infinity. Since $y = \ln b$, we see that the limit in question is indeed ∞.) That is,

$$\int_1^\infty \frac{1}{x}\,dx = \infty.$$

Definition 24.3.4. The integral $\int_a^\infty f(x)\,dx$ is called an **improper integral**. [1] When the integral $\int_a^\infty f(x)\,dx$ is finite, we say that the improper integral converges. Otherwise, we say it diverges.

[1] We will see other kinds of improper integrals in class.

Remark 24.3.5. Computing an improper integral combines many skills: You first have to be able to compute antiderivative F of the integrand f. Second, you need to be able to write the expression $F(b) - F(a)$ as a function of b. Lastly, you need to know how to take limits (as b approaches ∞).

For next time, I expect you to be able to say whether the following improper integrals converge or diverge. When an improper integral converges, I expect you to be able to state what number $\int_a^\infty f(x)\,dx$ is.

(a) $\int_1^\infty \ln x\,dx$.

(b) $\int_0^\infty e^{-x}\,dx$.

(c) $\int_3^\infty \frac{1}{x^3}\,dx$.

(d) $\int_2^\infty x^2\,dx$.