Lecture 25

From rates to amounts (applications of integration)

We have seen (through the fundamental theorem of calculus) that integration involves finding an antiderivative. This is a mathematical statement.

Today, we'll discuss *physical* and other *applied* interpretations of integrals.

Example 25.0.1. The speed of a car, at time t seconds, is given by

$$f(t) = 25 - (t-4)^2.$$

The units of f are in meters per second. (For example, at time $t = 0$, the car has speed equal to 9 meters per second. At time $t = 2$, the car has speed equal to 21 meters per second.)

Question: How far does the car travel between $t = 1$ and $t = 6$?

If the speed were *not* changing, you could use the formula

$$\text{distance} = \text{speed} \times \text{time}.$$

Unfortunately, speed is changing. So how do you solve the above problem?

Well, perhaps we could approximate. For example, instead of trying to solve the problem of how far the car traveled between $t = 1$ and $t = 6$, could we solve the problem of how far the car may have traveled between $t = 1$ and $t = 1.1$? (We're only considering a small portion of the time traveled.)

Because the time interval is now so small (it is only 0.1 seconds long!) we might expect that—even though the speed of the car is changing between $t = 1$ and $t = 1.1$—the speed of the car doesn't change *that much*. So if we pretend that the speed of the car is constant and use our naive formula of distance = speed × time, our naive formula's answer shouldn't be so far off the actual distance traveled. So, if we know that the car is going at $f(1)$ meters per second at time $t = 1$, our estimate for the distance traveled between time $t = 1$ and $t = 1.1$ could be given by

$$\begin{aligned} &\text{(estimated distance traveled} && (25.1)\\ &\text{between } t = 1 \text{ and } t = 1.1) = \text{(speed at } t = 1) \times \text{(time elapsed)} && (25.2)\\ &= f(1) \times (1.1 - 1) && (25.3)\\ &= f(1) \times 0.1. && (25.4) \end{aligned}$$

Now, we could then ask, how far did the car travel between $t = 1.1$ and $t = 1.2$? We can use $f(1.1)$ as an estimate for the speed of the car during this interval, again hope that pretending the speed is constant (i.e., non-changing) won't hurt us too bad, and calculate

$$\begin{aligned} &\text{(estimated distance traveled} && (25.5)\\ &\text{between } t = 1.1 \text{ and } t = 1.2) = \text{(speed at } t = 1.1) \times \text{(time elapsed)} && (25.6)\\ &= f(1.1) \times (1.2. - 1.1) && (25.7)\\ &= f(1.1) \times 0.1. && (25.8) \end{aligned}$$

We could do this for a bunch of small intervals, until we have estimated along the interval $[5.9, 6]$. (That means we'd have dealth with 60 small intervals total, each of width 0.1.) Adding up all our estimates, we can then hope to have a good estimate for total distance traveled. All in all, this "adding up" is a summation we can write as follows:

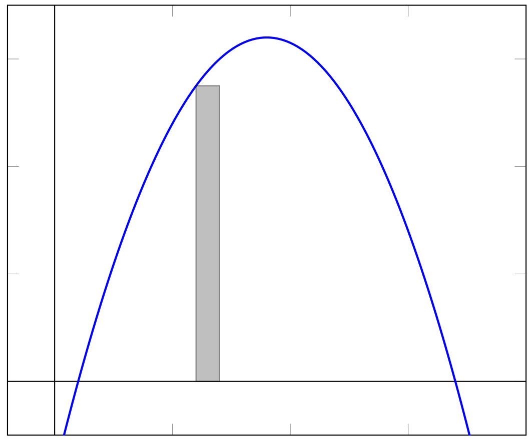
$$\text{Our estimate} = \sum_{i=1}^{60} (f(t_i) \times 0.1) \tag{25.9}$$

where x_i is given by $1 + 0.1i$. I hope this looks familiar—it looks like a Riemann sum!

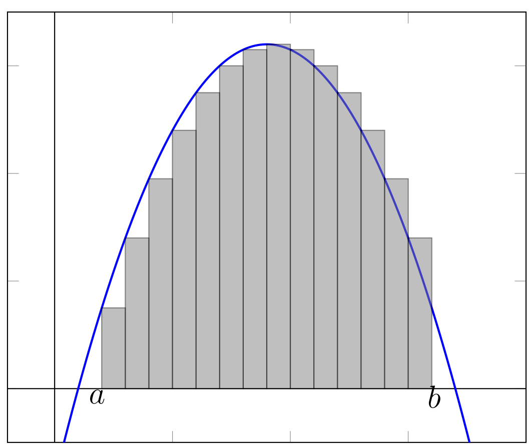
Indeed, we can graphically interpret what we are doing. The expression

$$\text{distance} = \text{speed} \times \text{time traveled}$$

can be interpreted as the *area* of a rectangle with height given by the speed, and base length given by the amount of time traveled.



(Pictured above is a rectangle—it has some height given by $f(t_i)$, and width given by δt_i, the amount of time traveled over a short interval.) Thus, our estimate (the summation in (25.9)) has a geometric interpretation as the sum of areas of all the rectangles pictured below:



And, of course, we expect our estimate to be better the more rectangles we take. But what does it mean to take more and more rectangles in our estimate? This is the *definition* of the integral! (Remember, the integral is defined to be a limit of Riemann sums as the number of rectangles n goes to infinity.)

Take-away: To solve the problem, we take the integral of $f(t)$ from $t = 1$ to $t = 6$.

I won't do out the math of taking the integral of $f(t)$ and producing the numerical answer; but I do want you to take away this conceptual nugget:

The integral of a rate of change gives the total change.

Example 25.0.2. Here are some examples of this concept in practice, along with where these ideas can be applied:

(a) (Physics/everyday travel)If $f(t)$ is the speed of an object at time t, then $\int_a^b f(t)dt$ tells us how far the object has moved between time a and time b.

(b) (Epidemiology/our lives during Covid-19) If $f(t)$ tells us the amount of new infections of a virus per day, then $\int_a^b f(t)dt$ tells us how many new infections have emerged between time a and time b (where time is in units of days).

(c) (Fuel efficiency/our daily lives) If $f(x)$ is a function telling us how many gallons of gas a car uses per unit distance, then $\int_a^b f(x)dx$ tells us how many gallons of gas a car uses to go from point a to point b.

(d) (Environmental science) If $f(p)$ measures the concentration of radiation per mile, then $\int_a^b f(p)dp$ measures the total amount of radiation between points a and b.

(e) (Biology/Urban planning) If $g(t)$ measures the rate of change of population at time t, then $\int_a^b g(t)dt$ tells us how much the population has changed between times a and b.

(f) (Engineering) If $\phi(t)$ measures, in liters per second, how quickly water is leaking from a bucket, then $\int_a^b \phi(t)dt$ tells us, in liters, how much water leaked from the bucket between times a and b.

(g) (Chemistry and endocrinology) If $c(t)$ measures the reaction rate of a chemical (for example, as measured in concentration per unit time), then $\int_a^b c(t)dt$ tells us the total change in the concentration of a chemical between times a and b.

Exercise 25.0.3. In the following, write down the integral. You do not need to solve for a numerical answer.

(a) In the absence of air resistance, it is known that a falling object on earth has velocity given by

$$v(t) = -9.8t$$

where $t = 0$ is the moment the object begins to fall (and has no speed).[1] How far does a falling object travel between times $t = 1$ and $t = 4$?

[1] The negative sign is present because we interpret a downward movement as negative.

If you are curious and want to perform computations: You drop a coin from a bridge, and your stopwatch tells you it took four seconds for the coin to hit the water. Assuming no air resistance, how high is your bridge from the water?

(b) Suppose that you invest 100 dollars into an index fund at time $t = 0$. It is commonly (and optimistically) estimated that the rate of growth of an index fund is given by

$$g(t) = 100 \ln(1.05) e^{(\ln 1.05)t}$$

where t is measured in years, and g is in dollars per year. How much money will be in your index fund at $t = 10$?

If you are curious and want to perform computations: How much money did you gain between $t = 0$ and $t = 5$? How much money did you gain between $t = 5$ and $t = 10$? Which of these two periods gave you more growth in terms of raw dollars?

(c) The acceleration $a(t)$ of a car is given by $a(t) = 12.2t$. (Acceleration is the rate of change of speed—note that this is *not* the same thing as speed!) How much speed did the car gain between $t = 1$ and $t = 2$?

Conceptual question: Can you tell how far the car traveled between $t = 1$ and $t = 2$ based on the information given? Why or why not?

25.1 Preparation for next time

I expect you to be able to do every problem in Exercise 25.0.3, *including the italicized portions*. This means I expect you to be able to compute all the relevant integrals, and answer the conceptual questions, too.