Lecture 26

The area of a circle

Throughout this class, I have tried to emphasize that you cannot just know that something is true; you must know *why* something is true. (This is why we prove things.)

Today, you are going to prove—in groups—that the area of a circle of radius R is given by πR^2.

26.1

Stare at the following integral:

$$\int_0^R \sqrt{R^2 - x^2}dx.$$

Why does this compute *one-fourth* of the area of a circle of radius R?

(Hint: A circle of radius R, centered at the origin, is described by the equation $y^2 + x^2 = R^2$. How does this hint help?)

26.2

Here is a strange substitution. Let's set

$$x = R\cos\theta$$

where θ is a new variable. Using this substitution, re-write the integral from the previous problem in a form that uses only the θ variable, and where no x variable appears. Your final answer should look something like

$$\int_a^b \text{something}\, d\theta.$$

Make sure you specify what a and b are. They should look a little funny!

(Hint: Is there a geometric interpretation for the substitution $x = R\cos\theta$?)

26.3

Evaluate the integral you've written in the previous problem. It is a difficult integral to evaluate at first glance. It may help to remember the following trig identities:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \qquad \cos(\theta)^2 + \sin(\theta)^2 = 1.$$

(Hint: It is hard to integrate $\sin^2$ or $\cos^2$; can you turn your integral into something that involves only things like $\cos(2\theta)$?)

26.4

How does your answer from the previous problem help you prove that the area of a circle of radius R is πR^2?

26.5 Preparation for next time

The speed of a moving object is described, in meters per second, by the following function:

$$v(t) = 3t^2.$$

(a) How far does the object travel between $t = 1$ and $t = 5$?

(b) On *average*, how fast was the object traveling during that time interval?

Based on last class, you know how to do the first part of this problem. The answer is given by

$$\int_1^5 3t^2\,dt = t^3\Big|_1^5 \tag{26.1}$$

$$= 5^3 - 1^3 \tag{26.2}$$

$$= 125 - 1 \tag{26.3}$$

$$= 124. \tag{26.4}$$

Now, how do you describe the average velocity? Well, the object traveled 124 meters. And it did so over 4 seconds. So on average, the object was traveling

$$\frac{124}{4} = 31 \text{ meters per second.}$$

Let's dissect this fraction a little bit. The numerator, 124, came from an integral. The denominator, 4, came from the *length* of the interval over which we took the integral. If we weren't given concrete values like 1 and 5, another way to write this fraction would have been

$$\frac{\int_a^b v(t)\,dt}{b-a}.$$

Why did we take perfectly good numbers and make them abstract, into some fraction with a lot of variables? Because these variables tell us how we should think of "average" quantities in other situations!

Definition 26.5.1. Let $f(x)$ be any function. Then the *average value of f* over the interval $[a, b]$ is the number given by

$$\frac{\int_a^b f(x)\,dx}{b-a}.$$

In words, the average value over an interval is obtained by taking the integral of f over the interval, and then dividing by the length of the interval.

Example 26.5.2. In terms of puppies per day, the local pound is reporting that puppies are being adopted at a rate given by the following function:

$$r(t) = 2^t. \tag{26.5}$$

For example, at time $t = 3$, puppies were being adopted at a rate of $r(3) = 2^3 = 8$ puppies per day. Between $t = 0$ and $t = 4$, on average, how many puppies were being adopted per day?

Solution: We must compute the integral $\int_0^4 r(t)dt$, and then divide by $4 - 0 = 4$.

Let's compute the integral:

$$\begin{aligned}
\int_0^4 2^t\, dt &= \int_0^4 e^{(\ln 2)t}\, dt && (26.6)\\
&= \frac{1}{\ln 2} e^{(\ln 2)t}|_0^4 && (26.7)\\
&= \frac{1}{\ln 2}(2^4 - 2^0) && (26.8)\\
&= \frac{1}{\ln 2}(16 - 1) && (26.9)\\
&= \frac{1}{\ln 2} \cdot 15 && (26.10)\\
&= \frac{15}{\ln 2}. && (26.11)
\end{aligned}$$

Then the average is given by

$$\frac{\int_0^4 r(t)\, dt}{4 - 0} = \frac{\frac{15}{\ln 2}}{4} = \frac{15}{4 \ln 2}.$$

You can plug this into a calculator in case you're curious about the puppies—they were adopted at a rate of about 5.41 puppies per day.

Example 26.5.3. What is the average height of the function $\cos(x)$ along the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$?

We must compute

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx$$

and then divide by the length of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. First, the integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx = \sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \quad (26.12)$$

$$= \sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}) \quad (26.13)$$

$$= 1 - (-1) \quad (26.14)$$

$$= 2. \quad (26.15)$$

The length of the interval, on the other hand, is

$$\frac{\pi}{2} - (-\frac{\pi}{2}) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Hence the average height is given by

$$\frac{2}{\pi}.$$

For next time, I expect you to be able to do the following:

(a) Find the average height of the function $\sin(x)$ along the interval $[0, \frac{\pi}{2}]$.

(b) A rocket's velocity is given by the function $v(t) = t^3$. What is the average speed of the rock between times $t = 2$ and $t = 4$?

(c) A virus is infecting people at a rate of e^{3t} new people per day. On average, how many new people per day were being affected over times $t = 0$ and $t = 3$?