Lecture 27

Average values

For today, you learned the following definition:

Definition 27.0.1. Let f be a function that is continuous on the interval $[a, b]$. Then the *average value* of f along the interval $[a, b]$ is defined to be

$$\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right).$$

In words, the average value is the integral from a to b, divided by the length of the interval.

Today we'll mostly be practicing computing average values. Before we dive in, let me give some idea as to why this number deserves to be called an average.

Suppose that we were given n numbers, say $a_1, \ldots, a_n$, and we wanted to compute their average. We would say

$$\text{average}(a_1, \ldots, a_n) = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

So, suppose that we have our function f, and we want to *approximate* what we mean by the average value. One thing we could do is to choose n numbers $x_1, \ldots, x_n$ in the interval $[a, b]$, and compute

$$\frac{1}{n}\left(f(x_1) + f(x_2) + \ldots + f(x_n)\right). \tag{27.1}$$

This would compute the average value of the numbers $f(x_1), \ldots, f(x_n)$; this might give some approximation to what we mean by the average value of f itself.

Well, we're used to the idea of dividing $[a, b]$ into n *equal* intervals. (We did this for Riemann sums.) So if we choose to do so, the number n takes on a different interpretation. Recall that the width of a rectangle in a Riemann sum is given by

$$\Delta x = x_i - x_{i-1} = \frac{(b-a)}{n}.$$

Hence,

$$n = \frac{b-a}{\Delta x}.$$

Substituting this into the approximation for average in (27.1), we find

$$\text{average} \approx \frac{1}{n}\left(f(x_1) + f(x_2) + \ldots + f(x_n)\right) \tag{27.2}$$

$$= \frac{1}{\frac{b-a}{\Delta x}}\left(f(x_1) + f(x_2) + \ldots + f(x_n)\right) \tag{27.3}$$

$$= \frac{\Delta x}{b-a}\left(f(x_1) + f(x_2) + \ldots + f(x_n)\right) \tag{27.4}$$

$$= \frac{1}{b-a}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right). \tag{27.5}$$

$$\tag{27.6}$$

Of course, as we make n larger (so that we choose more x_i), whatever we mean by average should approximate the "actual" average better and better. So let's take the limit as n goes to infinity—remember, that's the *definition* of the integral:

$$\text{average} = \lim_{n\to\infty} \frac{1}{b-a}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) \tag{27.7}$$

$$= \frac{1}{b-a}\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) \tag{27.8}$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{27.9}$$

So that was an argument to try and convince you that Definition 27.0.1 is a reasonable one. Now you can go practice in groups.

27.1

Using the mean value theorem (remember that?) convince me of the following:

If f is continuous on the interval $[a, b]$, then there is some number c between a and b so that $f(c)$ equals the average value of f over $[a, b]$.

27.2

For each f below, find the average value of f along the given interval.

(a) $f(x) = 4x - x^2$ along the interval $[1, 5]$.

(b) $f(t) = te^{-t^2}$ along the interval $[2, 3]$

(c) $f(\theta) = \sec^2(\theta/2)$ along the interval $[0, \pi/2]$.

27.3

Find the average value of $f(x) = \cos(\sqrt{x})$ along $[0, \pi^2/4]$. You may want to use u substitution, and then integration by parts.