Lecture 30

Logistic functions and logistic distributions, I

30.1 Review of last time

Last time, we started talking about one way to model the spread of a virus.[1] There were three steps to this:

1. Positing how the function $P(t)$—which tells us the number of infected people at time t—should behave. The first thing we started with was, "on average, every infected person infects k new people per unit time."

2. Writing down an *equation* that turned this intuition into mathematics. We saw:
$$\frac{dP}{dt} = kP(t)$$
of, equivalently, $P' = kP$. In words, this equation says that the rate of change of $P(t)$ is *proportional* to $P(t)$ itself; that is, how quickly the virus is spreading is proportional to how many people have the virus. (The more people that have it in a given moment, the more *new* people are about to get it.)

   This kind of equation is called a *differential equation* because it involves the function's *derivatives*. Note that the natural, intuitive equation that arose from step 1 isn't some explicit formula like $P(t) = t^2$, but some weird equation that involves derivatives. This happens *all the time* in the sciences; it's often easier to say what the derivative should do, rather than what the function is.

[1] It turned out that the mathematics for modeling this was *identical* to the mathematics modeling population growth, and growth of an investment, and radioactive decay!

3. The last step was a theorem: I just told you (without proof) that every function $P(t)$ satisfying the above differential equation is of the form

$$P(t) = Ae^{kt}$$

where A is some real number and k is the constant from the previous differential equation.

In fact, this is the process for almost all modeling of dynamical systems, with step 3 usually being the most difficult. Indeed, most interesting physical systems are *not* modeled by a nice equation like $P(t) = Ae^{kt}$. If you ever take a serious mathematical modeling class involving dynamical systems, you'll learn how to study systems without having a formula for their solutions.

30.2 What you read for this time

Exponential functions grow fast. A function of the form Ae^{kt} is called an exponential function. For example, 2^t is an exponential function (by setting $A = 1$ and $k = \ln 2$). In your reading for this time, you read that exponential functions grow *very quickly*—as t gets bigger, $P(t)$ gets *really* big, and fast.

Exponential functions grow faster when you change k. Moreover, this growth rate *depends on* k in a sensitive way, too. Just increasing k by a little can have huge consequences. You read that our everyday actions can influence k for the coronavirus, and it's important to take action early on. That's why the days, the weeks, and (in very uncoordinated and poorly led places like the United States) the months of delayed response can have disastrous public health consequences.

Exponential functions are only good at modeling early parts of an outbreak. Viruses can't spread indefinitely—for one thing, there are only so many human beings to infect. So obviously, Ae^{kt} (whose value will quickly surpass the population of human beings) can't be a super-accurate model for the later stages of an outbreak.

30.3 Today: The logistic function

We're going to see how to fix this last problem. We'll look for a function $P(t)$ that tells us how many people are infected at time t, but satisfying some different properties from last time.

Last time, I told you that exponential functions (those for which $P' = kP$) are good at modeling the beginnings of an outbreak. So, what should happen toward the "end" of an outbreak? Well, the number of infected people total should stabilize, as there are no new infections possible. In other words, we would expect P' to equal *zero* eventually.

Let's suppose that we want the growth rate of P to look like k as before, but we suppose that there's some "maximum" possible number of infected people, which we'll call K.

Then we're looking for a function $P(t)$ that satisfies the following properties:

(a) When $P(t)$ is small, we expect $P'(t)$ to be very close to $kP(t)$.

(b) When $P(t)$ is very close to K, we expect $P'(t)$ to be very close to zero.

At this stage, we have completed Step One of modeling our problem—we've identified some criteria we'd like P to satisfy.

Now, Step Two: Writing down a differential equation. Here is an equation that fits these two criteria.[2]

$$\frac{dP}{dt} = \frac{k}{K} P \cdot (K - P).$$

Let's see why this fits the above two criteria. The key is to look at the term in the parentheses.

(a) First, suppose that the value of P is small, so that $P(t) \approx 0$. Then the equation above looks like

$$\frac{dP}{dt} \approx \frac{k}{K} P \cdot (K - 0) \tag{30.1}$$

$$\approx \frac{k}{K} P \cdot K \tag{30.2}$$

$$\approx kP. \tag{30.3}$$

More rigorously, if t is a time at which $P(t)$ is small, then the slope $P'(t)$ at t is very close to $kP(t)$.

[2]Be careful: The k, K are constants, but P is a function. If you like, you can write in $P(t)$ each time you see P.

(b) Next, if the value of P is very close to K, so $P(t) \approx K$, then we have that

$$\frac{dP}{dt} \approx \frac{k}{K} P \cdot (K - K) \tag{30.4}$$
$$\approx \frac{k}{K} P \cdot 0 \tag{30.5}$$
$$\approx 0. \tag{30.6}$$

That is, if $P(t)$ is close to K at t,, then $P'(t)$ is close to 0 at t.

So we are looking for a function that satisfies the differential equation

$$\frac{dP}{dt} = \frac{k}{K} P \cdot (K - P).$$

As it turns out, we can find one:

Theorem 30.3.1. Let k and K be positive numbers. If $P(t)$ is any function satisfying the differential equation

$$\frac{dP}{dt} = \frac{k}{K} P \cdot (K - P),$$

such that P takes on only positive values, then $P(t)$ is given by a function

$$P(t) = \frac{K}{1 + e^{-k(t - t_0)}}.$$

Here, t_0 is the (unique) inflection point of $P(t)$.

Remark 30.3.2. If you ever take a differential equations class, you will learn how to find functions that solve particular differential equations.

Definition 30.3.3. Any function of the form

$$P(t) = \frac{K}{1 + e^{-k(t - t_0)}}.$$

(where k and K are positive numbers, while t_0 is any number) is called a *logistic function.*

The number k is called the *growth rate*, and K is called the *carrying capacity.*

Remark 30.3.4. By the way, the logistic differential equation is often written in the following (equivalent) form:

$$\frac{dP}{dt} = kP \cdot (1 - \frac{P}{K}).$$

Example 30.3.5. The following are all examples of logistic functions:

1. $\frac{1}{1+e^{-t}}$

2. $\frac{\pi}{1+e^{-3t}}$

3. $\frac{20000}{1+e^{-0.2(t-100)}}$

4. $\frac{3}{4+e^{-2(t-1)}}$

Make sure you can write down what the values of k, K, t_0 are in each of the examples above.

30.4 What the logistic function looks like

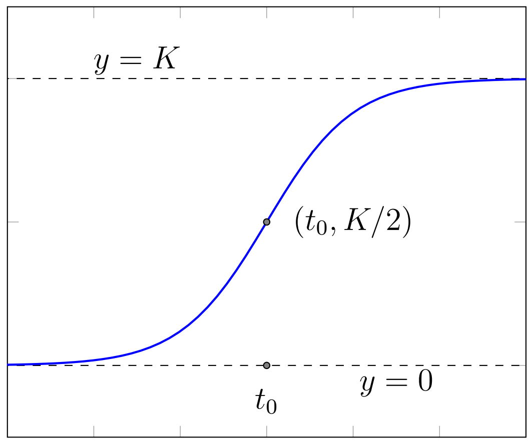
So, let's choose constants k, K, t_0, and study the logistic function

$$P(t) = \frac{K}{1 + e^{-k(t-t_0)}}.$$

Remark 30.4.1. We can use our *curve-sketching* techniques to draw this curve. As $t \to \infty$, we see that the denominator becomes $1 + e^{kt_0} \lim_{t\to\infty} e^{-kt}$; because the limit of e^{-t} is 0 as $t \to \infty$, we conclude that the denominator goes to 1. Thus, $P(t)$ has a horizontal asymptote of height K as t approaches infinity. Likewise, as t approaches negative infinity, the denominator approaches $1 + \infty$, so $P(t)$ has a horizontal asymptote of height 0 as $t \to -\infty$. Finally, every number involved is positive, and $e^{whatever}$ is positive, so we know P is always above the line $y = 0$.

Some calculations of derivatives and second derivatives show that $P'(t)$ is always positive (so the slope is always positive), and that there is a unique inflection point where P switches from concave down to concave up. Being careful about taking the second derivative, you'll see that the inflection point happens exactly at $t = t_0$.

Putting all this together, here is a graph of the logistic function:

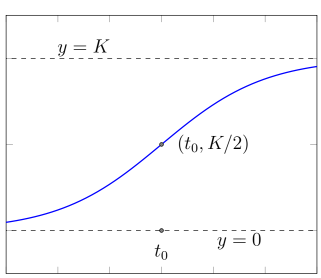


As you can see, the graph has two horizontal asymptotes of heights 0 and K. As you move forward in time (starting in the lower-left corner of the graph), the value of $P(t)$ begins close to 0, but then climbs, and then flattens: The value of $P(t)$ approaches K as we move forward in time.

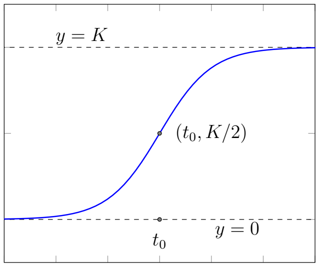
The graph has a unique inflection point, given at time $t = t_0$, as indicated.

Moreover, the value of $P(t)$ at the inflection point is $K/2$—i.e., half the carrying capacity.

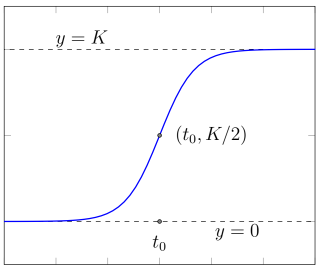
So we've seen the appearance of t_0 and of K in the above graph; but where does the growth rate k play a role? To see this, here are graphs of logistic functions with the same t_0 and K, but with varying k:

1. 

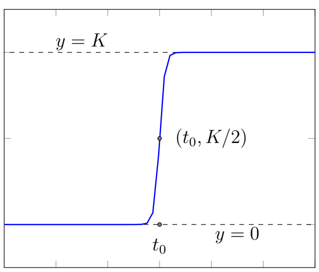
(k small)

2. 

(k medium-sized)

3. 

(k large)

4. 

(k huge)

As you can see, as k (the growth rate) is larger, the graph looks *steeper* near the inflection point. Thus, even if the carrying capacity is all the same (so that all the graphs approach K as t increases), we see a much more sudden jump toward the carrying capacity when k is larger.

In terms of the spread of viruses: Do you want to see the new cases of the virus all at once? (This is what happens when k is very large.) Or, would you rather see new cases at a more gradual pace? (This is what happens when k is small.) For example, do our hospitals want to see 1,000,000 new patients in a span of a day? Or do we want to see 1,000,000 new patients coming in over a span of twenty days?

30.5 Comparing to actual data

So, if we keep track of the number of cases of a new virus, do we get graphs that actually look like a logistic function? See the following pages.

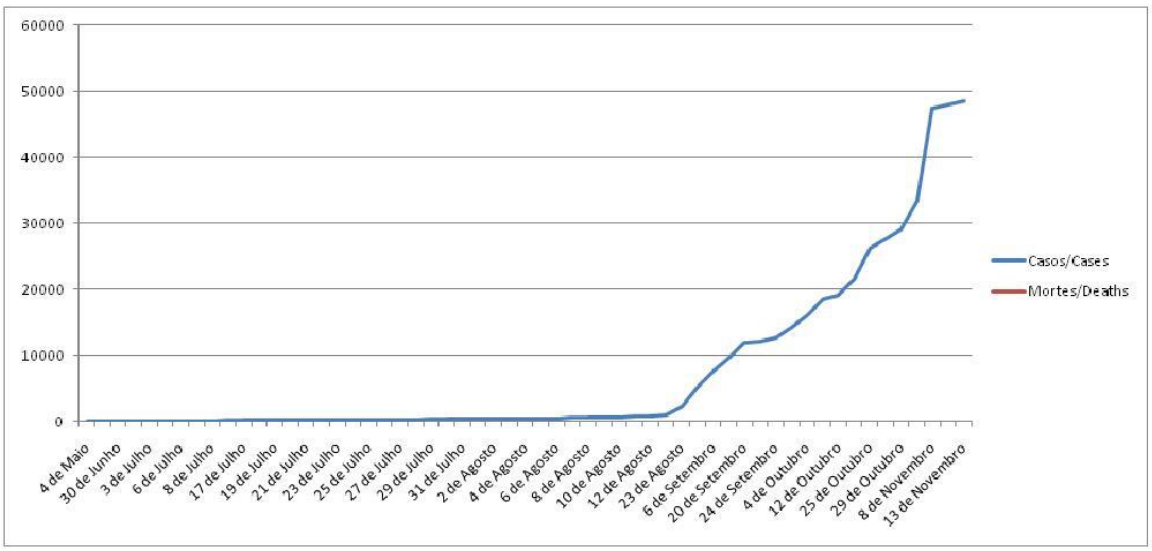


Figure 30.1: H1N1 cases in Portugal, 2009

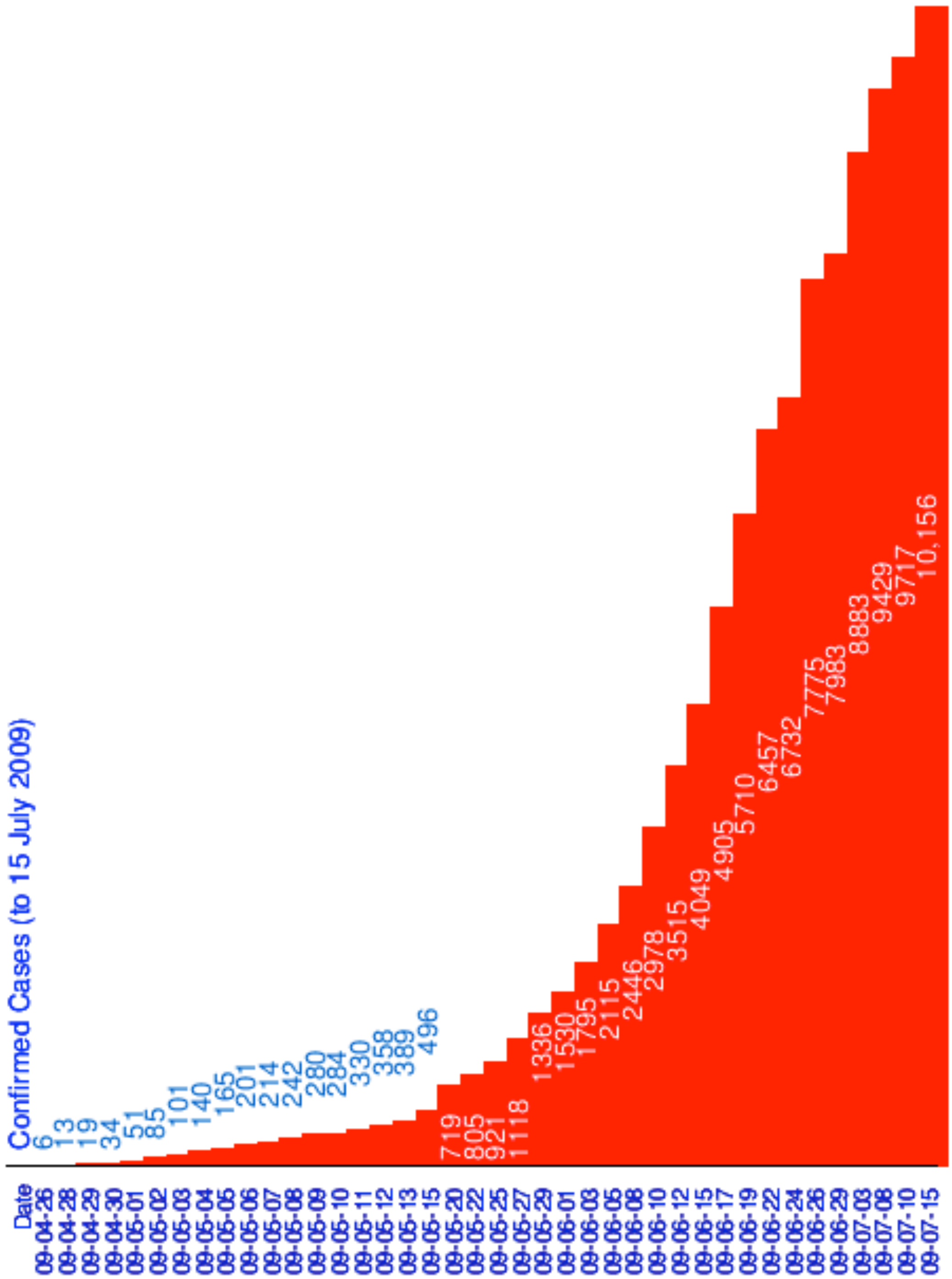


Figure 30.2: H1N1 cases in Canada, 2009

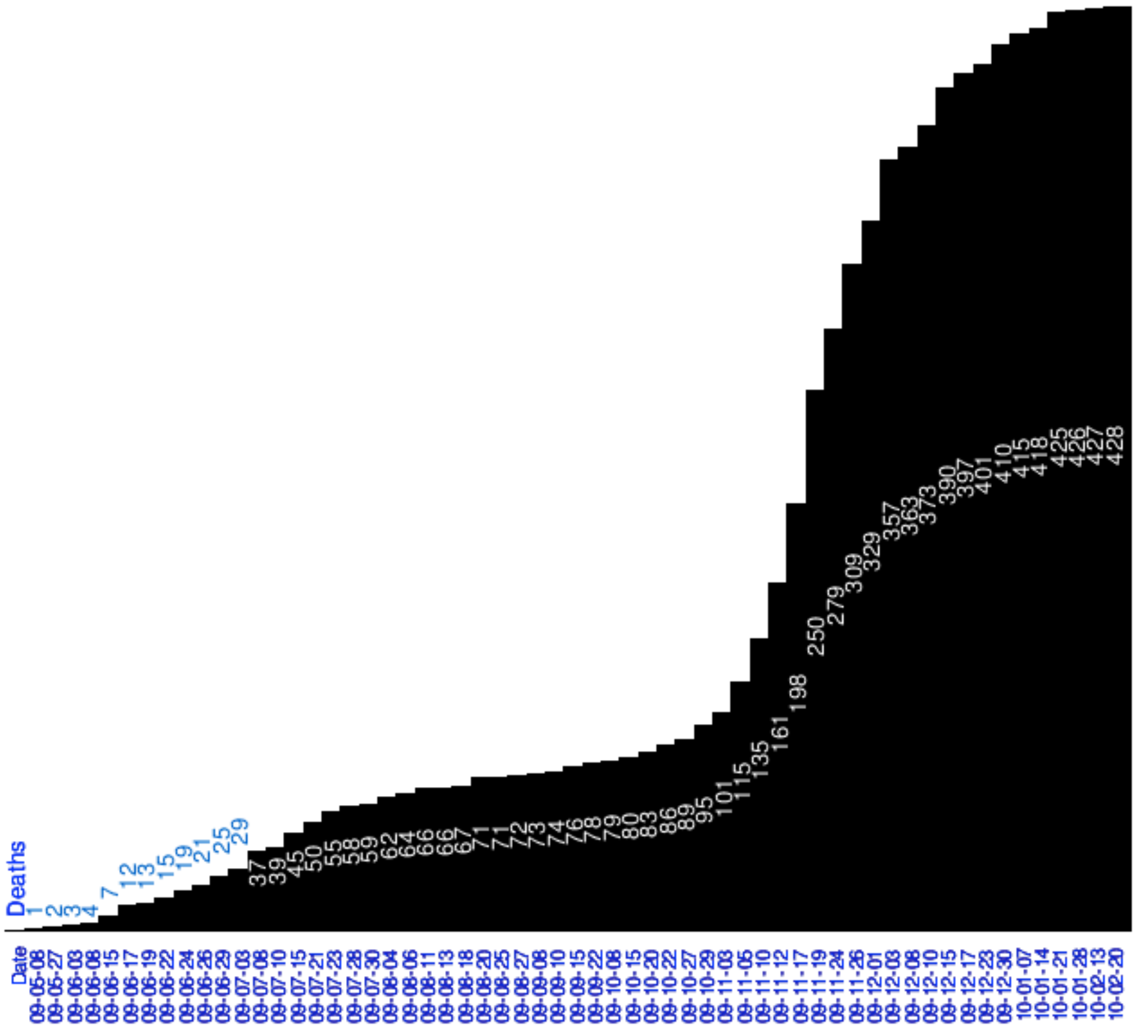


Figure 30.3: H1N1 deaths in Canada, 2009

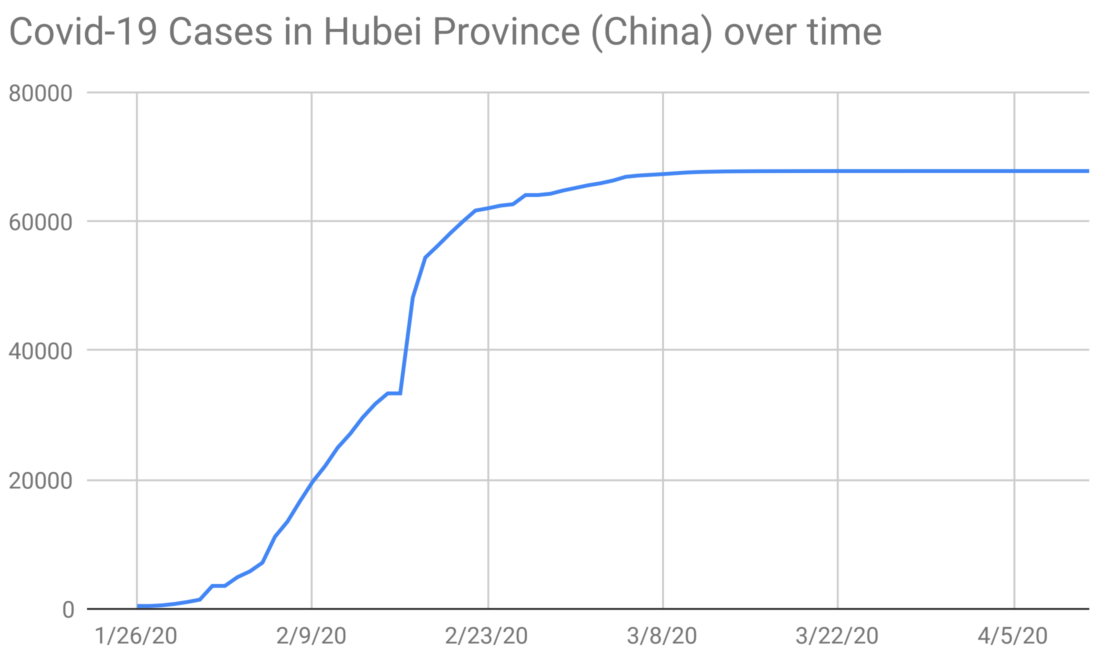


Figure 30.4: Covid-19 cases in Hubei Province (China), 2020

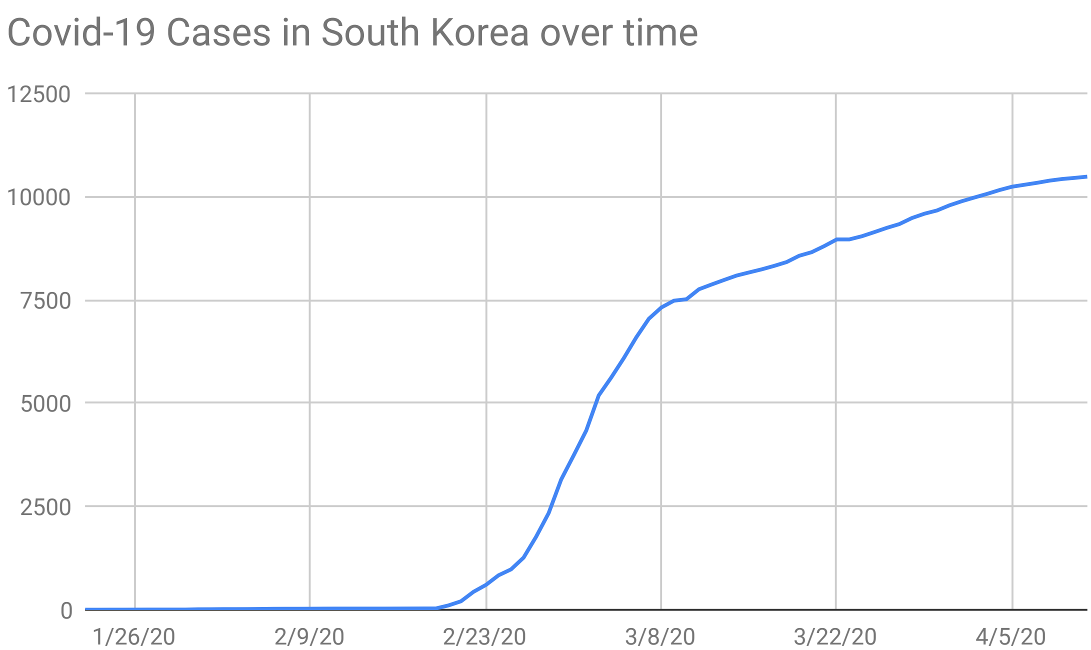


Figure 30.5: Covid-19 cases in South Korea, 2020

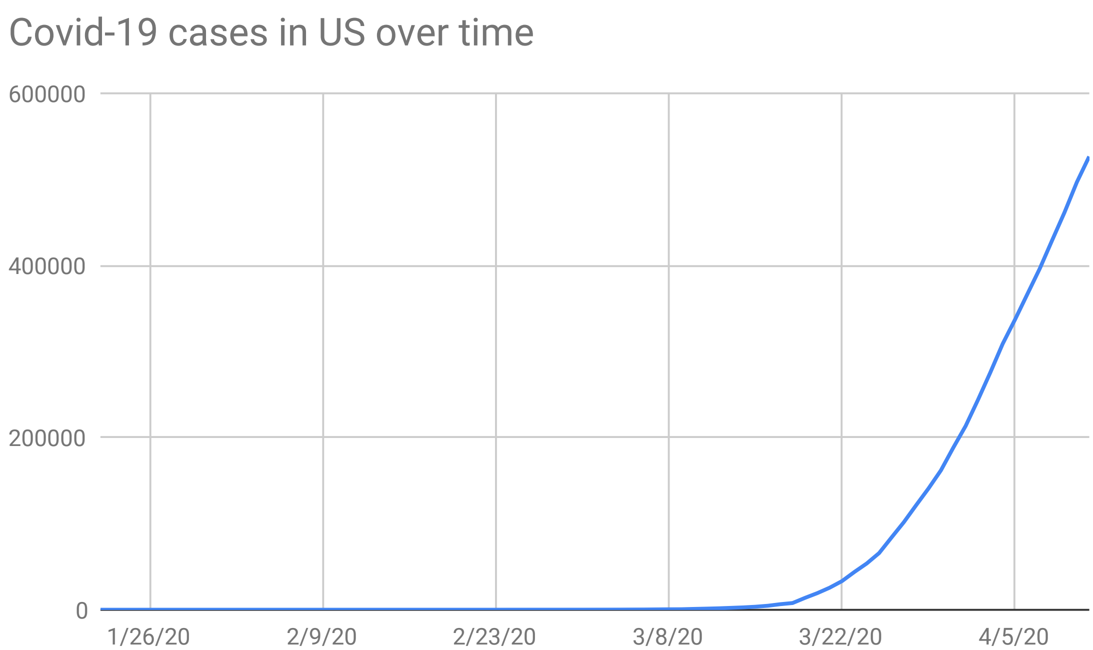


Figure 30.6: Covid-19 cases in USA, 2020

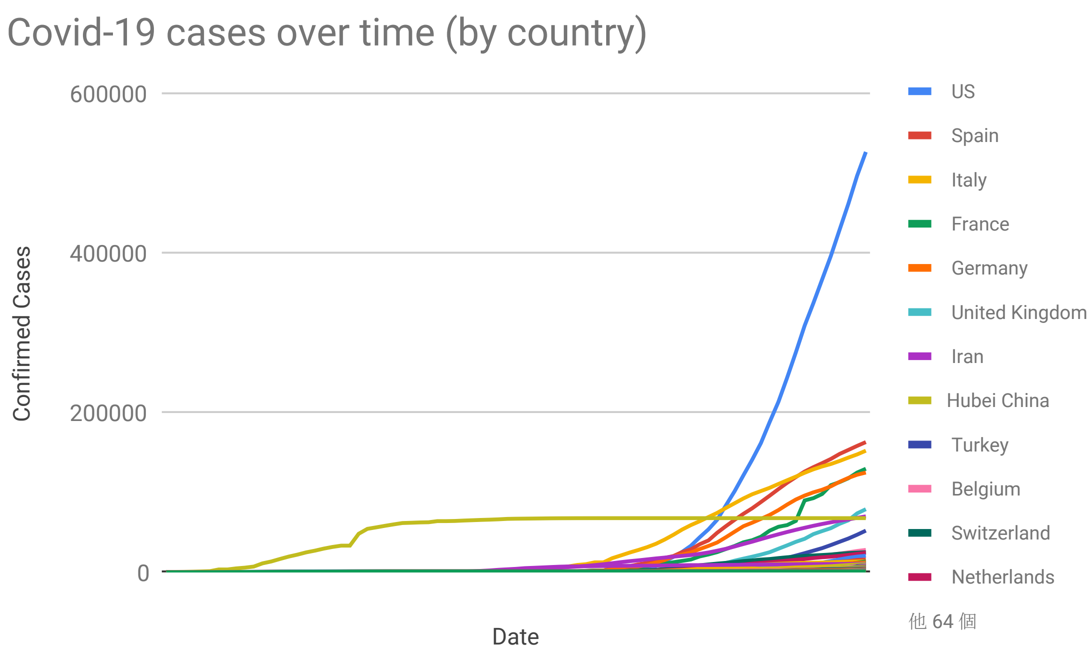


Figure 30.7: Covid-19 cases in various countries, 2020

30.6 Preparation for next time

30.6.1

Let $P(t)$ be a logistic function. Compute its derivative.

30.6.2

Verify that there is exactly one inflection point for the logistic function, occurring at $t = t_0$.

30.6.3

Instead of $P(t)$, consider the function that tells us how quickly $P(t)$ is changing at any give moment—that is, the function that tells us how quickly the number of infections is increasing at time t. At what value of t does this function achieve its maximum? What is this maximal rate, in terms of k, K and t_0?

30.6.4 (Plus One)

Note: The function $\arccos(x)$ is sometimes written $\cos^{-1}(x)$; this is how you may have learned it in precalculus.

(a) Draw the graph of $\cos(x)$ above the interval $[-2\pi, 2\pi]$.

(b) Draw the graph of $\arccos(x)$. Be very explicit about where the domain of your graph is.

(c) If x is a number between -1 and 1, what is $\cos(\arccos(x))$?

(d) What is $\arccos(\cos(x))$ when x is a number in the interval $[0, \pi]$? Does your answer change if x is not in this interval?