Lecture 31

Logistic functions and logistic distributions, II

31.1 Review of last time

Last time, we tried to improve on our model of virus outbreaks. We were motivated by trying to find a function $P(t)$ for which

1. When the value of $P(t)$ is small, $P(t)$ looks like an exponential function. That is,
$$P(t) \approx 0 \implies P' = kP.$$

2. When the value of $P(t)$ is near the carrying capacity K, $P(t)$ looks flat. That is,
$$P(t) \approx K \implies P' = 0.$$

One simple way to write these two requirements as a differential equation was

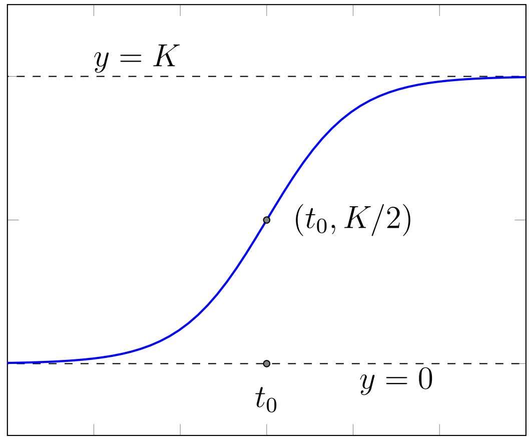
$$P'(t) = \frac{k}{K} P \cdot (K - P).$$

Then, I stated a theorem that any function P satisfying this differential equation must be given by the formula

$$P(t) = \frac{K}{1 + e^{-k(t-t_0)}}.$$

Any function of this form is called a *logistic* function.

The graph of the logistic function looks like this:



At time $t = t_0$, we have an inflection point. The slope at the inflection point is dictated by k.

We then compared the graph of the logistic function to actual data of confirmed cases of H1-N1, and of Covid-19. We saw that indeed, the graphs of actual data looked roughly like logistic functions.

Today, we're going to study the *derivative* of the logistic function.

31.2 Derivative of the logistic function

Let's begin with an exercise.

Exercise 31.2.1. Let

$$P(t) = \frac{\pi}{1+e^{-3(t-5)}}.$$

What is the slope of P at the inflection point?

Solution. We know the inflection point occurs at t_0, which in this case is 5. But how do we find the slope there? We have to take the derivative of P:

$$P'(t) = \left(\frac{\pi}{1+e^{-3(t-5)}}\right)' \tag{31.1}$$

$$= \frac{-\pi \cdot (-3)e^{-3(t-5)}}{(1+e^{-3(t-5)})^2} \tag{31.2}$$

$$= \frac{3\pi e^{-3(t-5)}}{(1+e^{-3(t-5)})^2}. \tag{31.3}$$

Plugging in the value $t = 5$, we find

$$P'(5) = \frac{3\pi e^{-3(5-5)}}{(1+e^{-3(5-5)})^2} \tag{31.4}$$

$$= \frac{3\pi}{(1+1)^2} \tag{31.5}$$

$$= \frac{3\pi}{4} \tag{31.6}$$

In fact, the value of $P'(t)$ at the inflection point is always given by the formula $kK/4$. To see this, we can carefully compute $P'(t)$ (which you are doing for homework anyway):

By the quotient rule, we have

$$P'(t) = \frac{(K)'(1+e^{-k(t-t_0)}) - K(1+e^{-k(t-t_0)})'}{(1+e^{-k(t-t_0)})^2}. \tag{31.7}$$

Using the chain rule, we see that

$$(1+e^{-k(t-t_0)})' = e^{-k(t-t_0)} \cdot (-k(t-t_0))' \tag{31.8}$$

$$= e^{-k(t-t_0)} \cdot (-k) \tag{31.9}$$

$$= -ke^{-k(t-t_0)}. \tag{31.10}$$

Plugging this in, we find

$$\frac{(K)'(1+e^{-k(t-t_0)}) - K(1+e^{-k(t-t_0)})'}{(1+e^{-k(t-t_0)})^2} = \frac{0\cdot(1+e^{-k(t-t_0)}) - K(-ke^{-k(t-t_0)})}{(1+e^{-k(t-t_0)})^2} \tag{31.11}$$

$$= \frac{kKe^{-k(t-t_0)}}{(1+e^{-k(t-t_0)})^2}. \tag{31.12}$$

Thus,

$$P'(t) = \frac{kKe^{-k(t-t_0)}}{(1+e^{-k(t-t_0)})^2}.$$

This is again a complicated-looking function at first glance. (Complicated is in the eye of the beholder—once you get used to this function, it won't seem so complicated.)

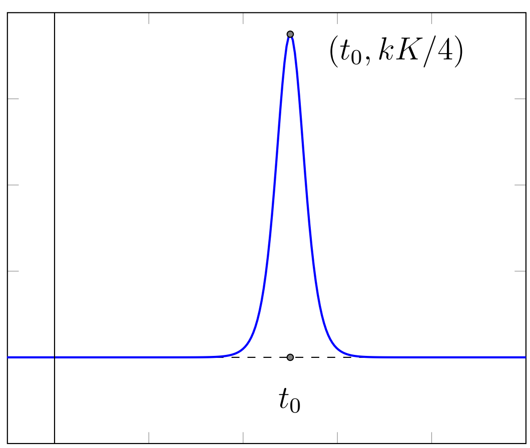
Definition 31.2.2. Any function

$$\frac{kKe^{-k(t-t_0)}}{(1+e^{-k(t-t_0)})^2}$$

is called a *logistic distribution.*

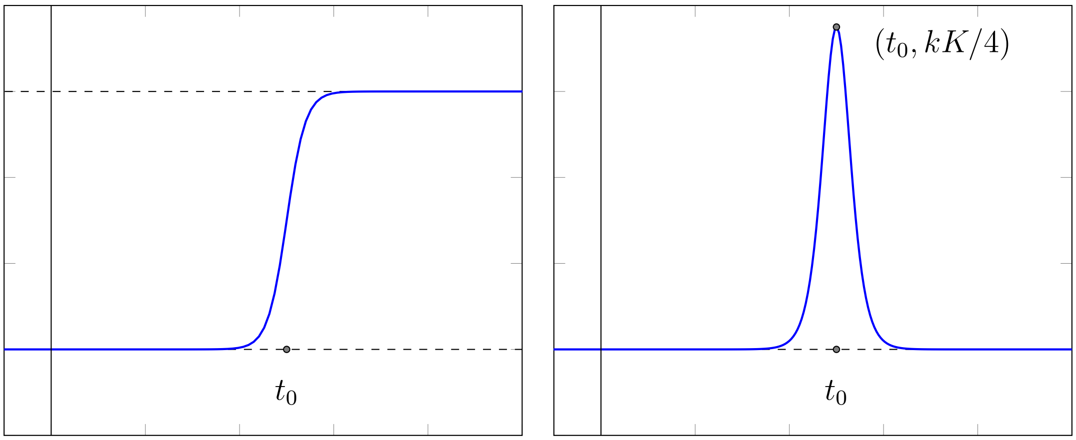
In other words, a logistic distribution function is the derivative of a logistic function.

Okay, let's come back down to earth. Here is a graph of a logistic distribution.



The function P' has a horizontal asymptote—at both $t=\infty$ and $t=-\infty$, the graph approaches the horizontal line given by $y=0$. The graph also has a unique local maximum, given at $t = t_0$, with height given by $kK/4$ (as you compute in your homework due tonight).

In case you are curious, here is the graph of both P and of P', side by side:



The spike on the right corresponds to the point of steepest slope on the lefthand graph, and the height of the spike is precisely the slope of that steepest point on the lefthand graph.

31.3 Comparing logistic distribution to real data

Let's get back to modeling the spread of viruses.

If our logistic function $P(t)$ is telling us how many people total have been infected by time t, then $P'(t)$ is the *rate of change* of infections. For example, if t is in units of days, the units of $P'(t)$ is in people per day. Concretely, $P'(t)$ tells us, at moment t, the rate at which the number of infected people is increasing (in units of people per day).

We rarely have a continuously changing $P'(t)$ that we can measure. What we often learn on the news is "Here are the total number of new cases that occurred the previous day." So, we can at least graph those numbers, and compare the graph to a logistic distribution. If our model is accurate, then we should expect the graphs of new infections per day to look similar.

Let's look at real data—do some of these look like a logistic distribution?

Remark 31.3.1 (Peaks)**.** The logistic distribution has a peak. This is not the time at which we see no more new cases; this is the time at which the number of new daily cases is maximal. You might imagine that this is the day when the most new patients become hospitalized. When you hear a lot of people on the news talking about the "peak" of the virus, they often mean the peak of the derivative, P'. So, even after the "peak," we will still see many new infections. The daily number of new infections will steadily decrease, however.

Remark 31.3.2. Graphs were made on Google Sheets, using raw data from Johns Hopkins University.

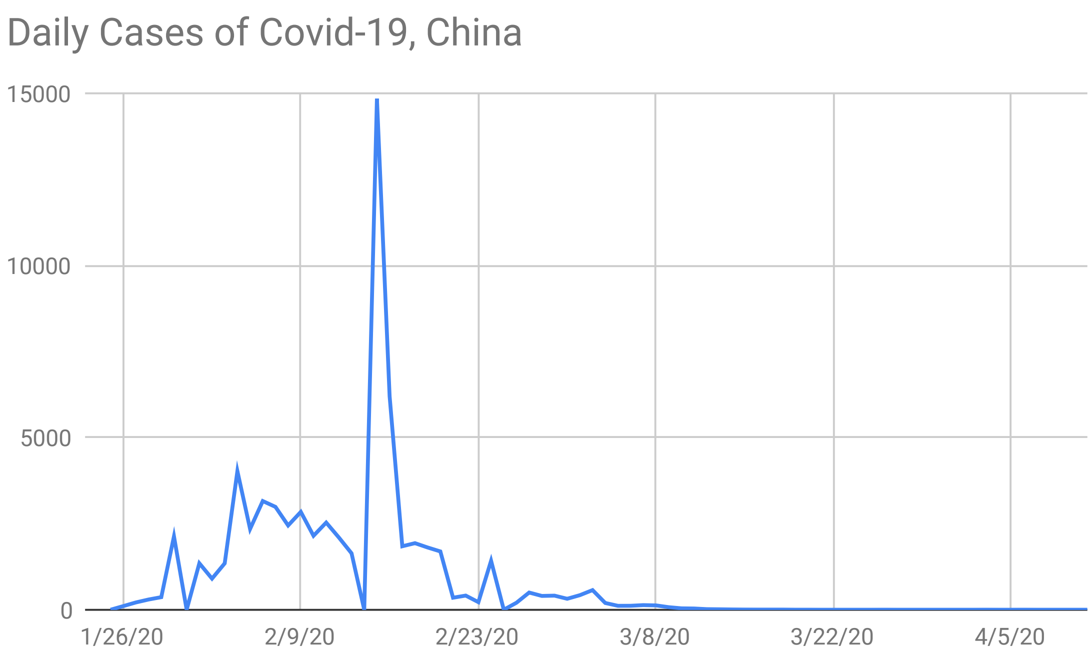

Daily Cases of Covid-19, China
15000
10000
5000
0
1/26/20
2/9/20
2/23/20
3/8/20
3/22/20
4/5/20

Figure 31.1:

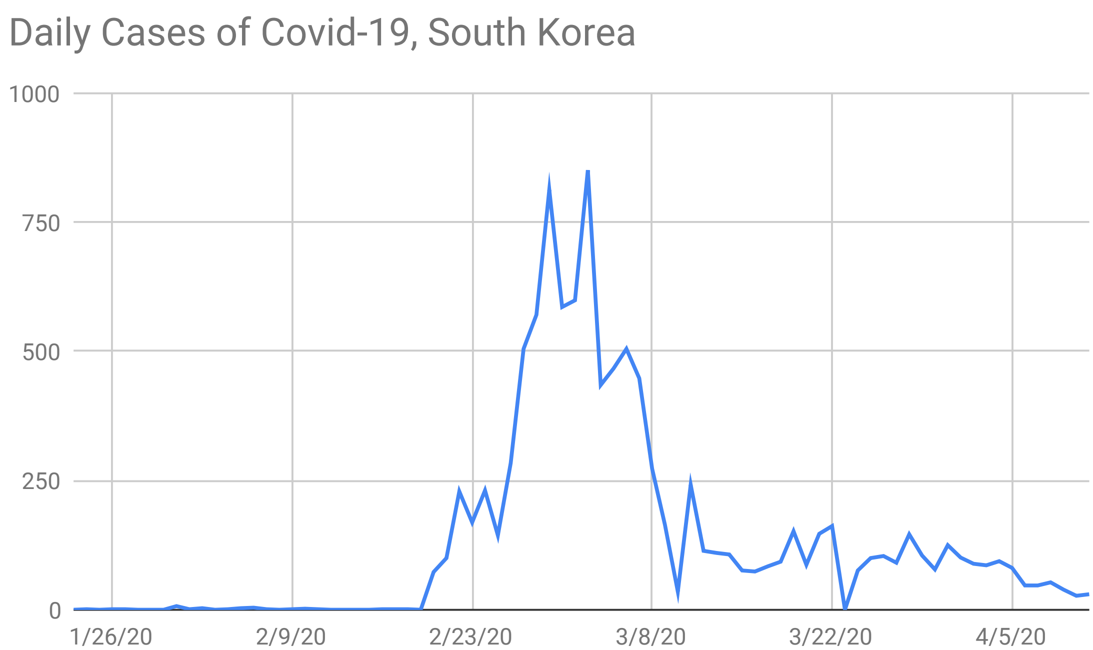

Daily Cases of Covid-19, South Korea
1000
750
500
250
0
1/26/20
2/9/20
2/23/20
3/8/20
3/22/20
4/5/20

Figure 31.2:

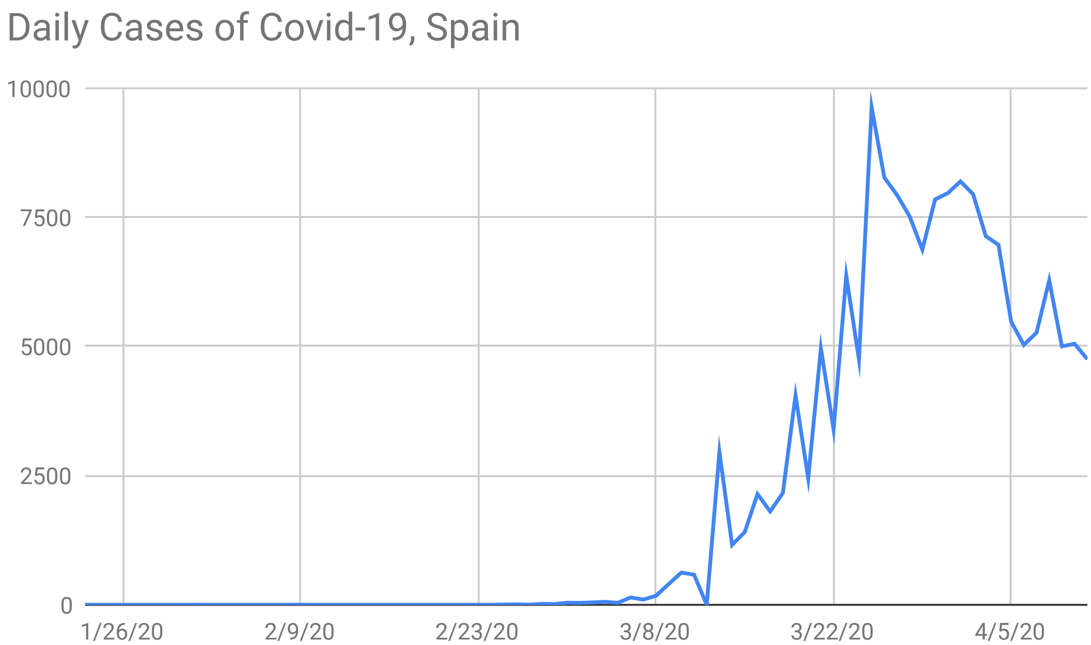

Daily Cases of Covid-19, Spain
10000
7500
5000
2500
0
1/26/20
2/9/20
2/23/20
3/8/20
3/22/20
4/5/20

Figure 31.3:

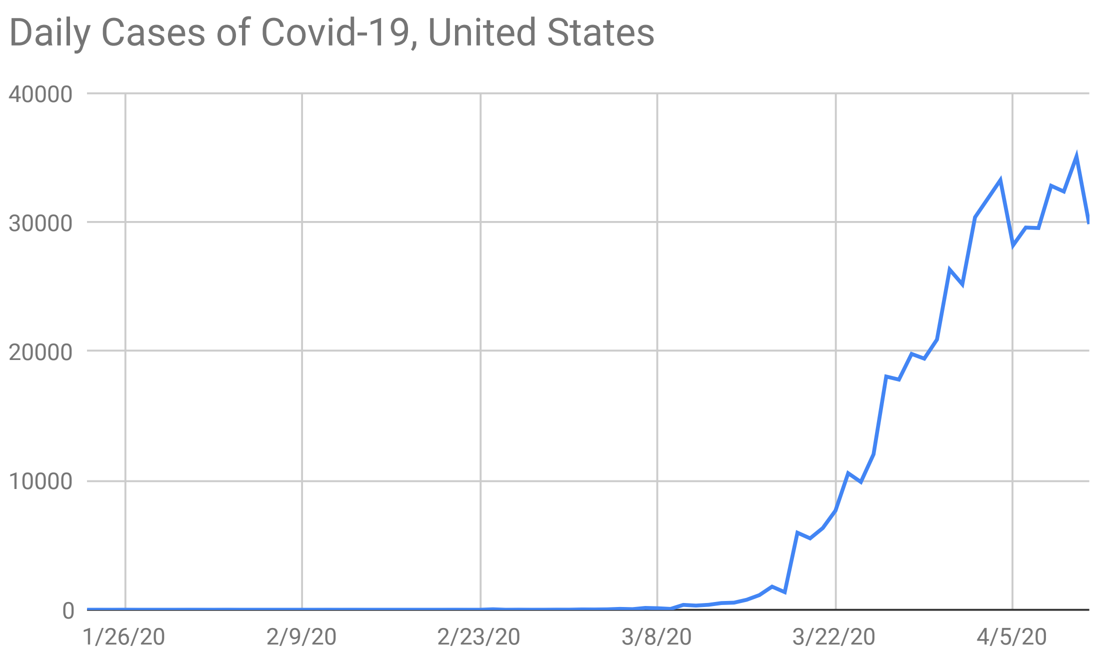

Daily Cases of Covid-19, United States
40000
30000
20000
10000
0
1/26/20
2/9/20
2/23/20
3/8/20
3/22/20
4/5/20

Figure 31.4:

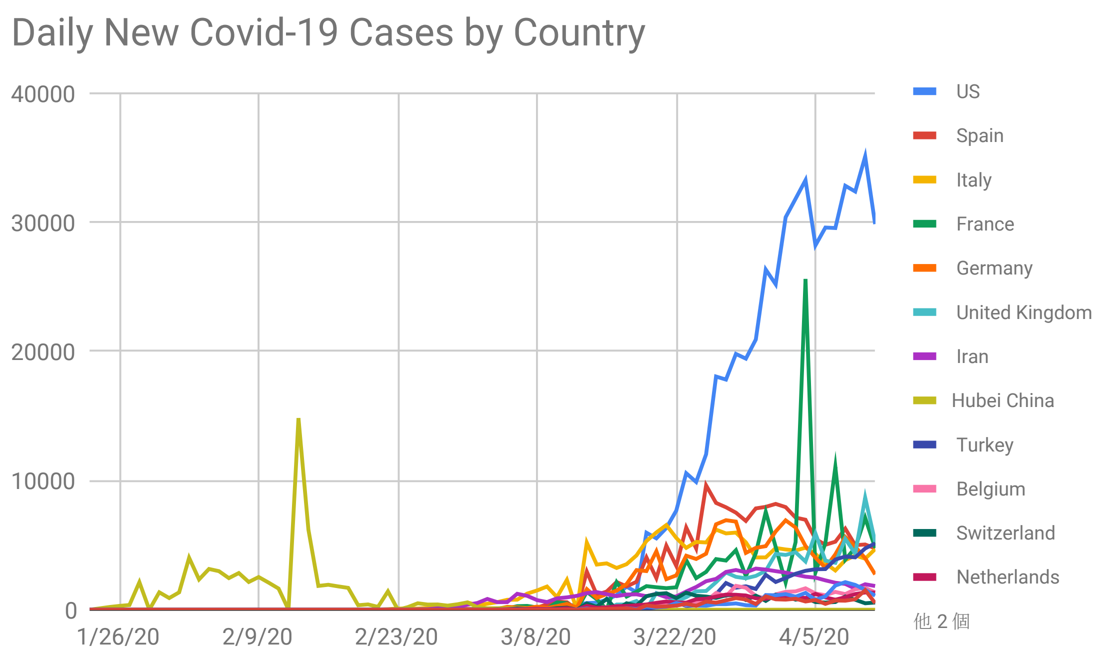


Figure 31.5: Covid-19 cases in various countries, 2020

31.4 Flattening the curve

You may heard the term "flatten the curve" in the media. We can model what this means mathematically.

Suppose that there's some carrying capacity K—an upper limit on how many people a virus can infect. Suppose also that today, at $t = 0$, there are exactly I people who have been infected. These are things out of our control. (We can't go back in time and change the number of people that are infected.) However, the growth rate k *is* under our control, because we as a population can choose to lower the rate of increase of the virus by (for example) practicing social distancing.

As it turns out, we can compute t_0 in terms of k, K, and I:

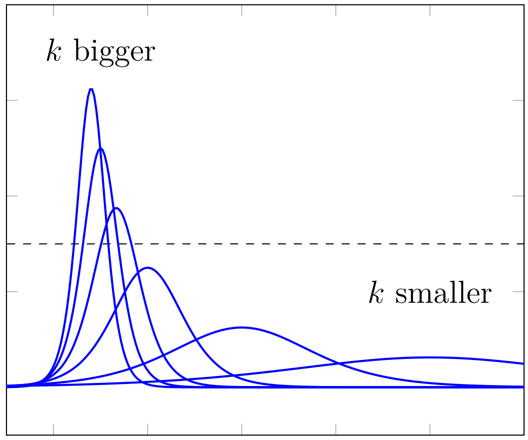
$$t_0 = \frac{\ln(\frac{K-I}{I})}{k}.$$

Note that as k gets smaller (so the infection rate is slowed), t_0 gets bigger. That is, we can *delay* the peak. Moreover, we've already seen that the peak of $P'(t)$ has value $kK/4$, so as we decrease k, we can also decrease the number of new patients that a hospital has to worry about. That's good! The upshot: The intuitive idea that social distancing reduces the rate of spread of a virus has (in this model) concrete outcomes, like delaying the peak, and shrinking the peak.[1]

So, fixing K and I, let's see what the graphs of the logistic distribution look like

[1]Shrinking the peak is good, because it's a lot easier for a hospital to worry about 500 new patients on the busiest day, as opposed to 10,000 new patients on the busiest day. However, the delay of the peak is arguably bad. Some people really want the stay-at-home orders to end sooner rather than later, for example; but any relaxation comes at the cost of making the peak taller, and over-burdening our hospitals. If our model is accurate, we see that it is mathematically impossible to shrink the peak without prolonging the outbreak.

for different values of k:



I've drawn a dashed horizontal line. Suppose our hospital system can *only* handle 100 new patients a day, and suppose the dashed horizontal line represents that threshold. (So, for example, for every day that our logistic distribution is above the line $y = 100$, our hospitals are working over capacity.) If we can reduce k, we can "flatten the curve" so that the peak occurs below the hospital capacity line of 100 new patients a day. You should think of every logistic distribution above as a preview of a possible future. Which curve do you think our future will be?

So the trade-off: We shrink the peak, but we prolong the outbreak. As a public health policy, most countries have chosen to try to shrink the peak at the risk of prolonging the outbreak. I am not an expert, but you can imagine the horrible consequences of a country's hospital system collapsing. For example, people who need care unrelated to Covid-19 would not receive care. And even after the outbreak is over, the damage to our hospital system (for example, due to casualties in our very small population of trained doctors and nurses) may take a long time to recover.

31.5 Preparation for next time

31.5.1

Let P be a logistic function. Suppose you know that $P(0) = I$, and that you are also given the values of K and k. Prove that

$$t_0 = \frac{\ln\left(\frac{K-I}{I}\right)}{k}$$

31.5.2 (Plus One)

Suppose a line has slope $m \neq 0$, and goes through the point (x_0, y_0). Find the value of x at which the line crosses the x-axis.