Lecture 32

Derivatives of inverse trigonometric functions

Today we're going to learn some new derivatives! Let's begin with some review. This is an exercise we've already seen in class:

Exercise 32.0.1. Using the fact that $e^{\ln(x)} = x$, find the derivative of $\ln(x)$. (I know you know what the derivative of ln is; I want you to be able to *prove* your answer.) Hint: You'll need to use the chain rule.

The reason that I want you to prove your answer: Now you can find new derivatives!

Exercise 32.0.2. Remember that $\sin^2\theta + \cos^2\theta = 1$. So, for example

$$\sqrt{1-\cos^2\theta} = \sin\theta \qquad \text{and} \qquad \sqrt{1-\sin^2\theta} = \cos\theta$$

whenever $\sin\theta \geq 0$ and $\cos\theta \geq 0$.

(a) Using the fact that $\sin(\arcsin(x)) = x$, find the derivative of $\arcsin(x)$.

(b) Using the fact that $\cos(\arccos(x)) = x$, find the derivative of $\arccos(x)$.

(c) Using the fact that $\tan(\arctan(x)) = x$, find the derivative of $\arctan(x)$.

Here are solutions to the exercises.

32.1 Derivative of arcsin

Note that $\arcsin(x)$ is defined when $x \in [-1, 1]$, with an output/range of $[-\pi/2, \pi/2]$. An important thing to note is that $\cos\theta \geq 0$ whenever $\theta \in [-\pi/2, \pi/2]$. As a result:

$$\sin(\arcsin(x)) = x \tag{32.1}$$

$$\cos(\arcsin(x)) \cdot (\arcsin(x))' = 1 \tag{32.2}$$

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} \tag{32.3}$$

$$= \frac{1}{\sqrt{1 - \sin^2 \arcsin(x)}} \tag{32.4}$$

$$= \frac{1}{\sqrt{1 - (\sin \arcsin(x))^2}} \tag{32.5}$$

$$= \frac{1}{\sqrt{1 - x^2}}. \tag{32.6}$$

Note that **even though** $\arcsin(x)$ **is continuous along** $[-1, 1]$**, its derivative only exists along** $(-1, 1)$**.** (That is, $\arcsin(x)$ is **not differentiable** at -1 and at 1.) Can you explain why using the graph of arcsin?

Also note that the derivative of arcsin has two vertical asymptotes, at $x = 1$ and $x = -1$. You can see this by taking the limit of $\frac{1}{\sqrt{1-x^2}}$ at $x = 1$ (approaching from the left) and the limit at $x = -1$ (approaching from the right):

$$\lim_{x\to 1^-} \frac{1}{\sqrt{1 - x^2}} = \infty \qquad \text{and} \qquad \lim_{x\to -1^+} \frac{1}{\sqrt{1 - x^2}} = \infty.$$

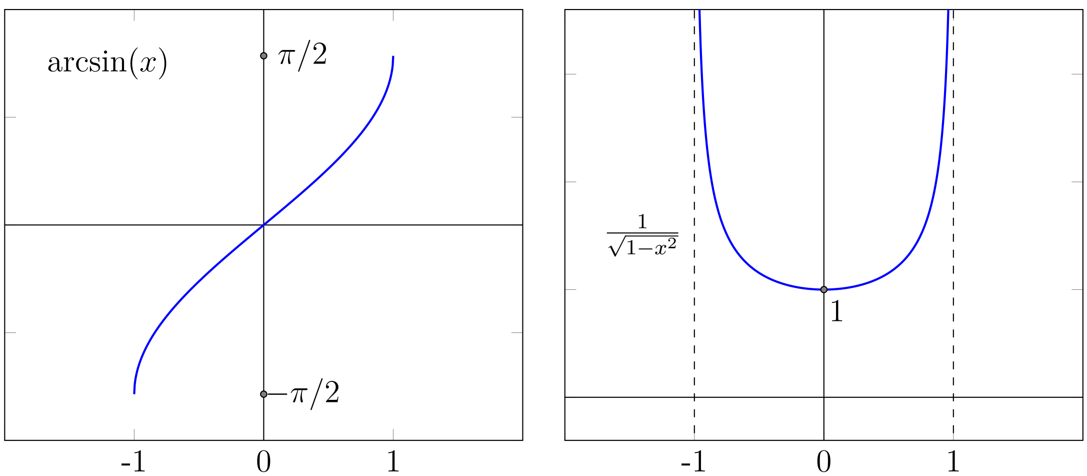

Figure 32.1: The graph of $\arcsin(x)$ on the left. The graph of its derivative on the right. Note the vertical asymptotes of the derivative.

32.2 Derivative of arccos

Now let's try arccos. This has a domain of $[-1, 1]$ and a range of $[0, \pi]$. On this interval, sin takes on values between 0 and 1.

$$\begin{aligned}
\cos(\arccos(x)) &= x && (32.7)\\
-\sin(\arccos(x)) \cdot (\arccos(x))' &= 1 && (32.8)\\
(\arccos(x))' &= \frac{1}{-\sin(\arccos(x))} && (32.9)\\
&= \frac{1}{-\sqrt{1-\cos^2 \arccos(x)}} && (32.10)\\
&= \frac{1}{-\sqrt{1-(\cos \arccos(x))^2}} && (32.11)\\
&= \frac{1}{-\sqrt{1-x^2}} && (32.12)\\
&= \frac{-1}{\sqrt{1-x^2}}. && (32.13)
\end{aligned}$$

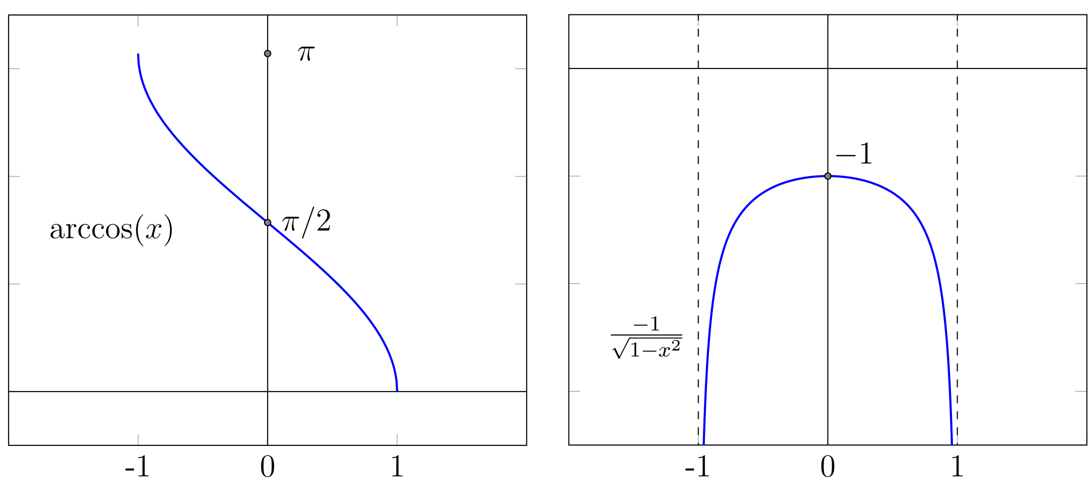

Figure 32.2: The graph of arccos(x) on the left. The graph of its derivative on the right.

32.3 Derivative of arctan

Finally, let's compute the derivative of arctan. This has a domain of $(-\infty, \infty)$ and a range $(-\pi/2, \pi/2)$.

We will make use of the following:

$$\begin{aligned}
\sec^2(\theta) &= \frac{1}{\cos^2(\theta)} && (32.14)\\
&= \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} && (32.15)\\
&= \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} && (32.16)\\
&= (\tan\theta)^2 + 1. && (32.17)
\end{aligned}$$

Here's the computation of the derivative:

$$\begin{aligned}
\tan(\arctan(x)) &= x && (32.18)\\
\sec^2(\arctan(x)) \cdot (\arctan(x))' &= 1. && (32.19)\\
(\arctan(x))' &= \frac{1}{\sec^2(\arctan(x))} && (32.20)\\
&= \frac{1}{(\tan(\arctan(x)))^2 + 1} && (32.21)\\
&= \frac{1}{x^2 + 1}. && (32.22)
\end{aligned}$$

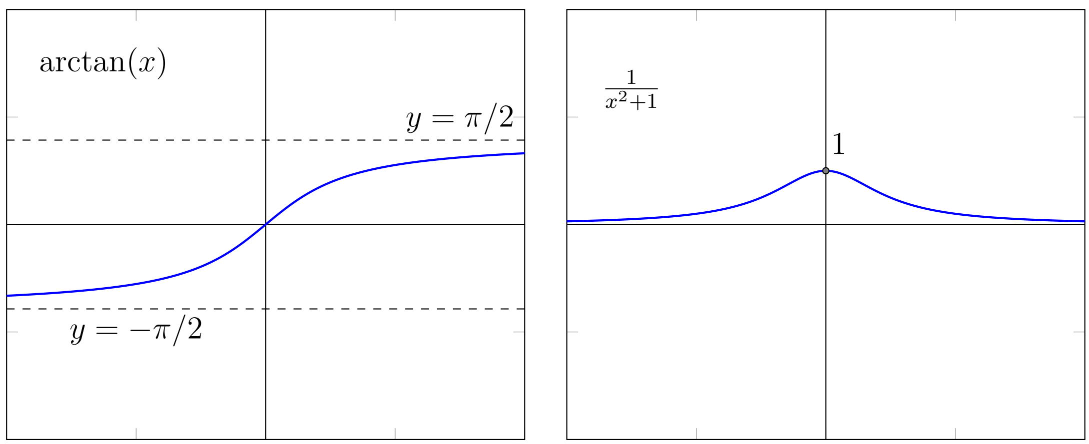

Figure 32.3: The graph of $\arctan(x)$ on the left. The graph of its derivative on the right.

32.4 Preparation for next time

32.4.1

(We did this in class, but make sure you could do this on a test!) Using the fact that $\sin(\arcsin(x)) = x$ and using a trig identity, prove that

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}.$$

32.4.2

(We did this in class, but make sure you could do this on a test!) Using the fact that $\tan(\arctan(x)) = x$ and using a trig identity, prove that

$$(\arctan(x))' = \frac{1}{x^2+1}.$$

32.4.3

Compute the following indefinite integrals:

(a)

$$\int \frac{1}{9x^2+1}\,dx$$

(b)

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx$$

(c)

$$\int \frac{1}{\sqrt{3-4x^2}}\,dx$$

32.4.4 (Plus One)

Let f be a function, and x_0 a real number. Find the equation of a line that goes through the point $(x_0, f(x_0))$, and is tangent to f at this point.