Lecture 34

Newton's Method, II

Last class, I introduced you to Newton's Method. This method takes a function f, and helps you find where f equals zero (i.e., where f has a root). I discussed how, more generally, this method allows you to take a function h and find out where h equals some value c.

Let's recall that Newton's Method works as follows:

- Step Zero is to choose a seed, x_0. This is just a choice of real number. I told you that the closer x_0 is to an actual root, the better Newton's Method will work.

- Step n has two parts. First, you find the tangent line to f at x_{n-1}. Then, you intersect the tangent line with the x-axis to obtain a new number, x_n, given by the x-coordinate where the intersection occurs. This x_n is your nth approximation to the root.

We left off by seeing a formula for how to get the nth approximation from the $(n-1)$st approximation:

$$x_n = x_{n-1} - \frac{f(x_n)}{f'(x_n)}.$$

34.1 Discussion of Newton's Method

This section is an elaboration on the class notes from last class.

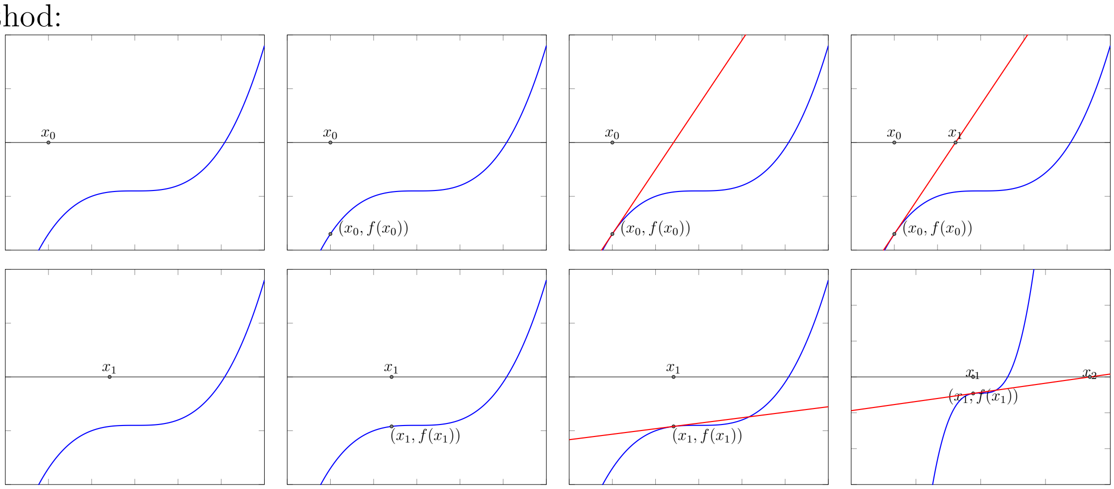
34.1.1 Newton's Method is recursive

Newton's Method is an example of a *recursive* algorithm. If you know what a for loop in computer programming is, you have seen recursive algorithms before. *Recursion* is a process by which we use a result from the previous step to compute something in the next step. Often, recursive algorithms repeat the same procedure every step; the only difference in each repetition is that the input may differ for each step.

For us, step n requires us to know x_{n-1} (that is, we need to know the output of the $(n-1)$st step). But the geometric process for taking x_{n-1} and outputting x_n is identical to all other steps of the method—you draw a tangent line, and find where it intersects the x-axis.

34.1.2 Newton's Method may fail

If you choose a bad seed x_0 Consider the following implementation of Newton's Method:



In the last square, we had to *zoom out* to find where the red line intersected the x-axis. Note that we don't see how the blue curve (the graph of f) behaves at $f(x_2)$. What if the blue curve exhibits the same kind of behavior as at x_1, forcing the tangent line to look kind of flat, so that our x_3 ends up even farther away from the root? And what if this happens over and over again, so that each x_n just gets us farther away?

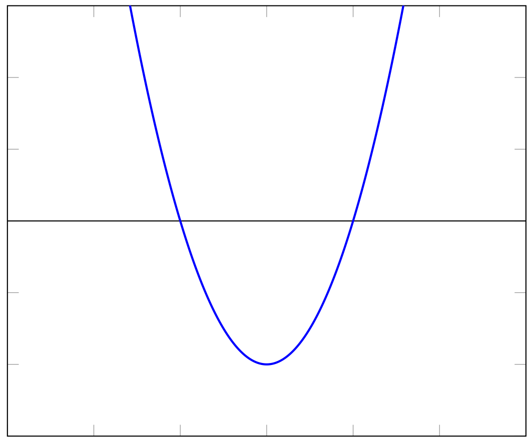
The reality is that *this could happen!* Indeed, for some functions, many choices of x_0 may force you into a terrible sequence of bad approximations. So, Newton's Method may fail. Keep this in mind.

Here are two other ways Newton's Method could fail.

If some x_n is a critical point It could happen that you choose x_0 so that some x_n is a critical point—that is, $f'(x_n) = 0$. Then the tangent line to f at x_n is flat, so it's parallel to the x-axis, meaning the tangent line will never intersect the x-axis! Thus we cannot go from the nth step to $(n+1)$st step. Newton's Method can't be implemented here, mainly because of an unlucky choice of seed. So, for example, you should try to *never* choose a critical point for your seed.

If f isn't differentiable Note that Newton's Method requires us to take a derivative at every step. So we want f to be a function that has all derivatives. Importantly, it turns out that we'll need f to be a function that is also differentiable at its root. As an example, if $f = \sqrt[5]{x}$ is the fifth root function, you'll have a hard time approximating the root because each time you get close to the root, Newton's Method will draw a very, very steep tangent line that will shoot you far away from the root.

If f has many roots Strictly speaking, this isn't a failure of Newton's Method. But Newton's Method *does not* tell you how many roots a function f has. (That is, it doesn't tell you how many times f itself intersects the x-axis.) For example, a lot of parabolas intersect the x-axis twice. (Take for example $f(x) = x^2 - 2x + -9$).[1]



Then, as a general rule, if your x_0 is close enough to the root on the right, Newton's Method will give you an approximate to the root on the right. If your x_0 is close enough to the root on the left, then Newton's Method will approximate the root on the left. Note that I wrote *close enough*, not *closer*. Indeed, for functions other than quadratic functions, being closer to one of the roots doesn't imply that you'll approximate that root.

So, just keep in mind that Newton's Method may approximate a root of a function, but you don't always know which root, nor how many roots a function f may have.

[1]By the way, by completing the square, we see that $f(x) = (x-1)^2 + 8$, so the roots of f are given by $1 \pm \sqrt{8}$. So if you know how to approximate the square root of 8, you'd know how to approximate the roots to f. Not every f allows you to make such simple reductions.

34.2 When, and why, does Newton's Method succeed?

Regardless, it turns out that *so long as one of your approximations x_n is close enough* to the actual root, then eventually, your approximations will become as close as you want to the actual root.

This is the result of a theorem that I won't mention any further, but the proof of the theorem uses something called the *contraction principle*, which you'll learn about if you ever take an advanced differential equations class or an advanced numerical analysis class. And, how close does one of your x_n need to be to the root? That depends on the values of the derivative of f at points nearby the root. In practice, it may be that you have a lot of information on these derivative values, or it may be that you have to fly blind.

Here is an argument as to why Newton's Method works.

First, let's look at the formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This is a beautiful formula, because it tells you what x_1 is for any x_0. In other words, this formula defines a function! Let's call the function g, so

$$g(x) = x - \frac{f(x)}{f'(x)}. \tag{34.1}$$

The function takes as input a possible seed, and outputs what x_1 of that seed is. Note that if x_0 is a seed for Newton's Method, then not only does $g(x_0) = x_1$, we also see that $g(g(x_0)) = x_2$, and $g(g(g(x_0))) = x_3$, and so forth.

For simplicity, let's assume that f has a root at $x = 0$. Then here is the big theorem:

Theorem 34.2.1. If $|g'(x)| < 1$ for all points x near the root, then $g(x)$ will always be closer to the root than x.

Let's see why this theorem should be true. If $|g'(x)| < 1$ for all x on some interval $[a, b]$, then you know that the integral of $g'(x)$ will have absolute value less than the integral of 1. But then for positive x,

$$|g(x)| \leq \int_{root}^{x} |g'(x)|\, dx \leq \int_{root}^{x} 1\, dx = x,$$

so we find that $|g(x)| \leq |x|$. (The same result holds for negative x.) Because we assumed the root is at the origin, $|x|$ is the distance of x from the root, and this inequality tells us that $|g(x)| < |x|$, meaning $g(x)$ is closer to the root than x originally was.

So the proof of the theorem relies crucially on the assumption that $|g'(x)| < 1$ near the root. Can we guarantee that this inequality holds? Well, let's try! The derivative of g is easy to calculate because we have an explicit formula for $g(x)$ (34.1). The derivative is

$$g'(x) = 1 = \frac{f'(x)f'(x) - f''(x)f(x)}{(f'(x))^2} \tag{34.2}$$

$$= 1 - 1 + \frac{f''(x)f(x)}{(f'(x))^2} \tag{34.3}$$

$$= \frac{f''(x)f(x)}{(f'(x))^2}. \tag{34.4}$$

Now suppose that $f'(root) \neq 0$—then $f'(x) \neq 0$ near the root.[2] Let m be the smallest value that $|f'(x)|$ takes on over some small interval near the root. Likewise, because $f''(x)$ is continuous, we can find some largest value M that $|f''(x)|$ takes on over this interval. Then $|g'(x)| \leq \frac{M}{m}f(x)$. The limit of $\frac{M}{m}f(x)$ as x approaches the root is 0, by definition of root! So, so long as x is close enough to the root, we can guarantee that $|g'(x)|$ is less than 1. (In fact, it'll be as small as we want if we make sure x is near enough to the root.)

Remark 34.2.2 (The need for higher derivatives of f)**.** If you're following along, you'll wonder what goes wrong when $f'(x) = 0$ at the root. Well, to compute

[2] We are assuming here that f' is continuous, so that nearby points have nearby values. It turns out f' is automatically continuous if f'' exists; we'll see this later.

$\lim_{x\to 0} g'(x)$, just apply L'Hopital's Rule to (34.4). Then we have

$$\begin{aligned}
\lim_{x\to 0} g'(x) &= \lim_{x\to 0} \frac{f''(x)f(x)}{(f'(x))^2} && (34.5)\\
&= \lim_{x\to 0} \frac{f'''(x)f(x) + f''(x)f'(x)}{2f'(x)f''(x)} \qquad \text{(L'Hopital)} && (34.6)\\
&= \lim_{x\to 0} \frac{f'''(x)f(x)}{2f'(x)f''(x)} + \frac{f''(x)f'(x)}{2f'(x)f''(x)} && (34.7)\\
&= \lim_{x\to 0} \frac{f'''(x)f(x)}{2f'(x)f''(x)} + \frac{1}{2} && (34.8)\\
&= \lim_{x\to 0} \frac{f^{(4)}(x)f(x) + f'''(x)f'(x)}{2f''(x)f''(x) + 2f'(x)f'''(x)} + \frac{1}{2} \qquad \text{(L'Hopital)} && (34.9)
\end{aligned}$$

Now if we assume that $f''(root) \neq 0$, the limit of this fraction can be computed, using the fact that $\lim_{x\to 0} f(x) = 0$ and$\lim_{x\to 0} f'(x) = 0$. For then, by the quotient law for limits,

$$\lim_{x\to 0} \frac{f^{(4)}(x)f(x) + f'''(x)f'(x)}{2f''(x)f''(x) + 2f'(x)f'''(x)} + \frac{1}{2} = \frac{0+0}{2f''(x)f''(x) + 0} = 0.$$

And indeed, if $\lim_{x\to root} g'(x) = 1/2$, then $|g'(x)| < 1$ for x close enough to the root. You'll note that to carry this through, even if $f'(root) = 0$, we needed to assume that $f''(root) \neq 0$. And, we had to take more derivatives (the fourth derivative showed up!). So to apply Newton's Method, we need to assume that some higher derivative of f doesn't equal zero at the root, and that f has enough higher derivatives beyond that.

34.3 Preparation for next time

34.3.1

Let $f(x) = x^2 - 3x - 1$.

(a) Using the quadratic formula, find an exact expression for the roots of f.

(b) Let $x_0 = 4$. Using Newton's Method three times (i.e., find x_3), approximate one of the roots of f. You may use a calculator; write down your x_3 to 6 decimal places. How does x_3 compare to the first few digits of your answers from (a)?

(c) Now suppose that instead, you took the seed $x_0 = -2$. Using the graph of f, explain why Newton's Method would produce an approximation to a different root than (b) approximated.

34.3.2 (Plus One) Intersecting curves

Consider the two curves $y = 2x - x^2$ and $y = -5$.

Find the two points where these two curves intersect.