Lecture 35

Areas between curves

The last couple weeks have focused on "applications" of the calculus tools we've learned. For example, we saw how to model a lot of interesting scientific problems using differential equations (through the examples of exponential growth and logistic growth). This week, we learned about Newton's Method.

Today we'll learn about another application of calculus tools—how to find the areas of a region bounded by various curves.

You already know that the integral measures the (signed) area between a graph and the x-axis. I say "signed" because any area below the x-axis is computed to be negative.
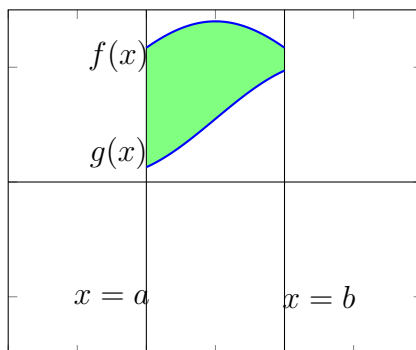
picture

Today, we'll learn how to take areas of different kinds of regions.

## 35.1   Areas between two vertical lines and two curves

Below is a region sandwiched between four curves:

1. The vertical line $x = a$,

2. The vertical line $x = b$,

3. A curve given by the graph of a function $f(x)$, and

4. A curve given by the graph of a function $g(x)$.

Note that the graph of f is "above" the graph of g in this picture. How would you find the area of the region?

Well, you could first find the area between the graph of f and the x-axis. Then, you could subtract off the area between the graph of g and the x-axis.
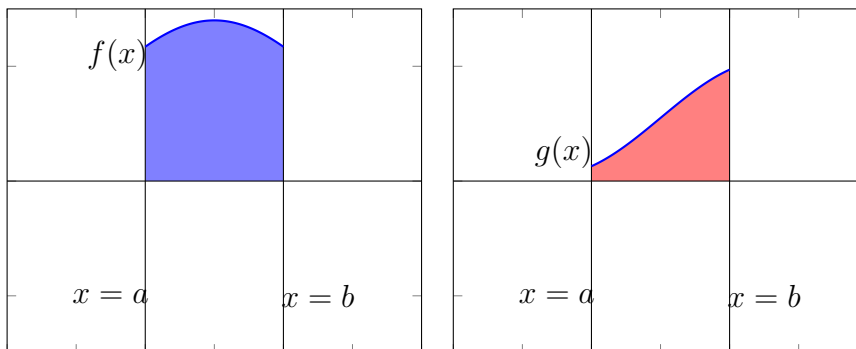
(If you remove the red region from the blue region, you end up with the green region from before.) This difference is the area between f and g. So we have

$$\text{Area} = \int_a^b f \, dx - \int_a^b g \, dx.$$

But by our rules for integrals, this righthand side can be rewritten; we obtain the following formula:

$$\text{Area} = \int_a^b f - g \, dx.$$

Exercise 35.1.1. Set up an integral that computes the area between the curves $y = \sin(x)$ and $y = 2 + x$ between the lines $x = 0$ and $x = \pi$.

Solution. Drawing the two curves, we see that between the vertical lines $x = 0$ and $x = \pi$, the graph of $\sin(x)$ is "below" the graph of $2 + x$. Hence to compute the area, we compute the integral of $2 + x - \sin(x)$. The region of integration is between $a = 0$ and $b = \pi$, so the integral is

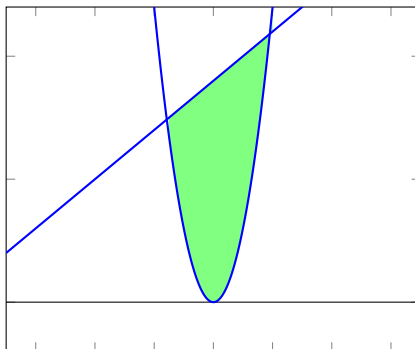$$\int_0^\pi 2 + x - \sin(x) \, dx.$$

□

35.2   Areas between two curves

You will also sometimes see problems likes this:

Exercise 35.2.1. Set up an integral computing the area between the curves $y = 3x^2$ and $y = x + 9$.

The above problem doesn't tell you a and b; but it assumes that you can find the region you're supposed to find the area of. Often, this requires you to draw a picture of the two curves in question:

Note that this region doesn't have "vertical walls" like the previous problem. Or, if you like, the vertical walls have been pinched to have height zero. However you interpret, the lefthand bound a is given by the smaller x-coordinate where the two curves intersect, while the righthand bound b is the larger x-coordinate where the two curves intersect.

So in solving these kinds of problems, you often need to find the coordinates of these intersection points.

Solution. We must find the intersections of $y = 3x^2$ and $y = x + 9$. Remember that if a point called (x, y) is in the intersection of these two functions, then the point

(x, y) satisfies both the equations $y = 3x^2$ and $y = x + 9$. So, we have that

$$3x^2 = x + 9 \tag{35.1}$$
$$3x^2 - x - 9 = 0 \tag{35.2}$$
$$x = \frac{1 \pm \sqrt{1 - 4(3)(-9)}}{2(3)} \tag{35.3}$$
$$= \frac{1 \pm \sqrt{1 + 108}}{6} \tag{35.4}$$
$$= \frac{1 \pm \sqrt{109}}{6}. \tag{35.5}$$

So $a = \frac{1 - \sqrt{109}}{6}$ and $b = \frac{1 + \sqrt{109}}{6}$. Moreover, looking at the picture (you are expected to be able to draw a sketch of both curves), we see that $y = x + 9$ is always above $y = 3x^2$. So the integral is given by

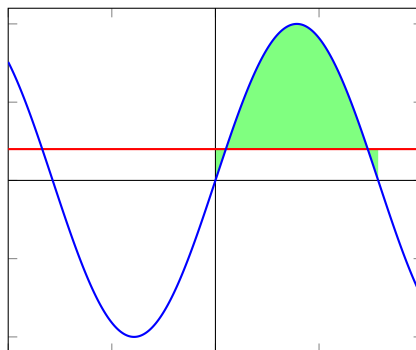$$\int_{\frac{1 - \sqrt{109}}{6}}^{\frac{1 + \sqrt{109}}{6}} x + 9 - 3x^2 \, dx.$$

$\square$

35.3   When top and bottom curves switch

So far, we have only talked about regions where a single function f is always "above" another function g. But some regions aren't defined this way.

Exercise 35.3.1. Find the area of the regions between $y = 0.2$ and $y = \sin(x)$ along the interval $[0, \pi]$.

It again helps to draw a picture to visualize the situation:

We have drawn $y = 0.2$ in red, $y = \sin(x)$ in blue, and the region in question in green. What we see is that there are roughly *three* regions between $y = 0.2$ and $y = \sin(x)$ along the interval from 0 to π. Indeed, the middle region is one where $y = 0.2$ is below the graph of $y = \sin(x)$, but in the other two regions, the horizontal line $y = 0.2$ is above the graph of $y = \sin(x)$.

So the best way to compute the total area of the regions is to compute the area of each of the three regions, then to add them together.

Solution. The lefthand region lives above the interval from 0 to $\arcsin(0.2)$. Here, the function $y = 0.2$ is above the curve $y = \sin(x)$, so the integral computing area is given by

$$\int_0^{\arcsin(0.2)} 0.2 - \sin(x)\, dx.$$

The middle region lives above the interval from $\arcsin(0.2)$ to $\pi - \arcsin(0.2)$. Here, the horizontal line $y = 0.2$ is *below* the curve $y = \sin(x)$, so the integral computing area is given by

$$\int_{\arcsin(0.2)}^{\pi - \arcsin(0.2)} \sin(x) - 0.2\, dx.$$

Finally, the righthand region lives above the interval from $\pi - \arcsin(0.2)$ to π. The integral now is given by

$$\int_{\pi-\arcsin(2.0)}^{\pi} 0.2 - \sin(x) \, dx.$$

Thus the total area is given by

$$\int_0^{\arcsin(0.2)} 0.2 - \sin(x) \, dx + \int_{\arcsin(0.2)}^{\pi-\arcsin(0.2)} \sin(x) - 0.2 \, dx + \int_{\pi-\arcsin(2.0)}^{\pi} 0.2 - \sin(x) \, dx.$$

$\square$

So be careful; sometimes the top and bottom functions can switch.

Remark 35.3.2. In fact, the briefest way to describe what's going on is to say that area is *always* given by the following integral:

$$\int_a^b |f - g| \, dx.$$

As you know, the absolute value sign is a function that you have to split up into a piecewise formula to evaluate. In the previous example, the reason three integrals popped up is because there are three regions in which the sign of $f - g$ alternate (and hence $|f - g|$ takes on different formulas).

35.4   When it's easiest to switch variables

There are some regions that are easier to define if we think of them as graphs of functions *of y*.

Exercise 35.4.1. Find the area of the region between the curves $x = \cos(y)$ and $x = \sin(y)$ between the lines $y = 0$ and $y = \pi/2$.
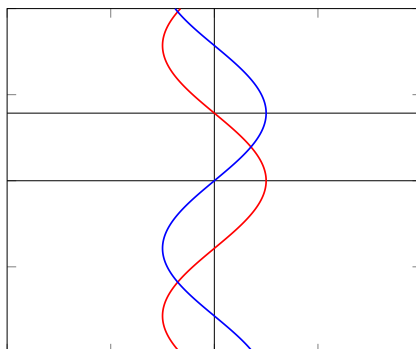
Solution. If you haven't seen the idea of swapping variables before, let me just tell you that you can also plot functions that depend on y as an input variable, and output x. Usually, $y = f(x)$ is a graph that satisfies the vertical line test. If instead we have $x = g(y)$, these graphs will satisfy the horizontal line test.

A fancy way to write the integral would be

$$\int_0^{\pi/2} \cos(y) - \sin(y)\, dy.$$

If it confuses you to integrate with respect to y, you can just use a different letter, like t:

$$\int_0^{\pi/2} \cos(t) - \sin(t)\, dt.$$

$\square$

35.5 Preparation for next time

35.5.1

Let f be a continuous function. I want you to compare

$$\int_a^b |f(x)|\, dx \qquad \text{and} \qquad |\int_a^b f(x)\, dx|.$$

(That is, compare the integral of the absolute value, to the absolute value of the integral.) When are these equal? When they are not equal, is one always larger than the other?

You can try drawing some examples of f to compare the two.

35.5.2 (Plus One) Inequalities

In the boxes below, draw $<, >, \leq, \geq,\ =$, or "Not enough information provided," whichever is most correct.

1. $-2 \,\square\, 3$

2. $-2 \,\square\, -3$

3. $|a + b| \,\square\, |a| + |b|$ (Here, a and b are real numbers.)

4. $|a - b| \,\square\, |a| + |b|$ (Here, a and b are real numbers.)

5. $|3 - a| \,\square\, 3 - |a|$. (Here, a is a real number.)