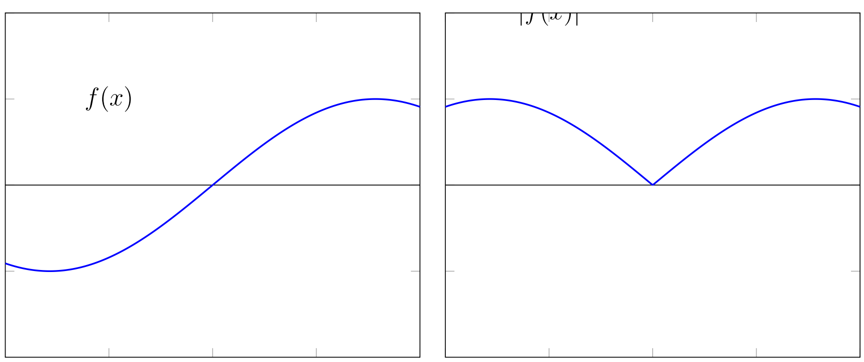
Lecture 36

How good are Taylor polynomial approximations?

For today, I had you reason out why (for a continuous function f) we have

$$\int_a^b |f(x)|\,dx \geq |\int_a^b f(x)\,dx|.$$

In case it helps, here are example pictures of what the functions $f(x)$ and $|f(x)|$ might look like:



How do their integrals compare?

We've already seen this inequality when verifying that Newton's Method works so long as the seed is close enough to a root. Today we'll use it again to see that we can actually *guarantee* that Taylor polynomials are good approximations!

36.1 Recollections on Taylor polynomials

Let's let f be a function that has derivatives, and second derivatives, and third derivatives, and so forth.[1] Choose also a real number a. Then we found a way to write down polynomials whose graphs looked a lot like the graphs of f, at least when we're close to a.

First, let's recall the calculus and the algebra. Writing down a Taylor polynomial involved the following steps:

1. Decide what degree Taylor polynomial you want. This was often in the prompt. Let's say this degree is 4.

2. Compute the derivatives of f all the way to the 4th derivative. This means you have to compute the functions $f(x), f'(x), f''(x), f'''(x)$, and $f^{(4)}(x)$.

3. Compute the *values* of these derivatives at a. So you have to compute the numbers $f(a), f'(a), f''(a), f'''(a), f^{(4)}(a)$.

4. Use reasoning (or a formula) to conclude that the polynomial

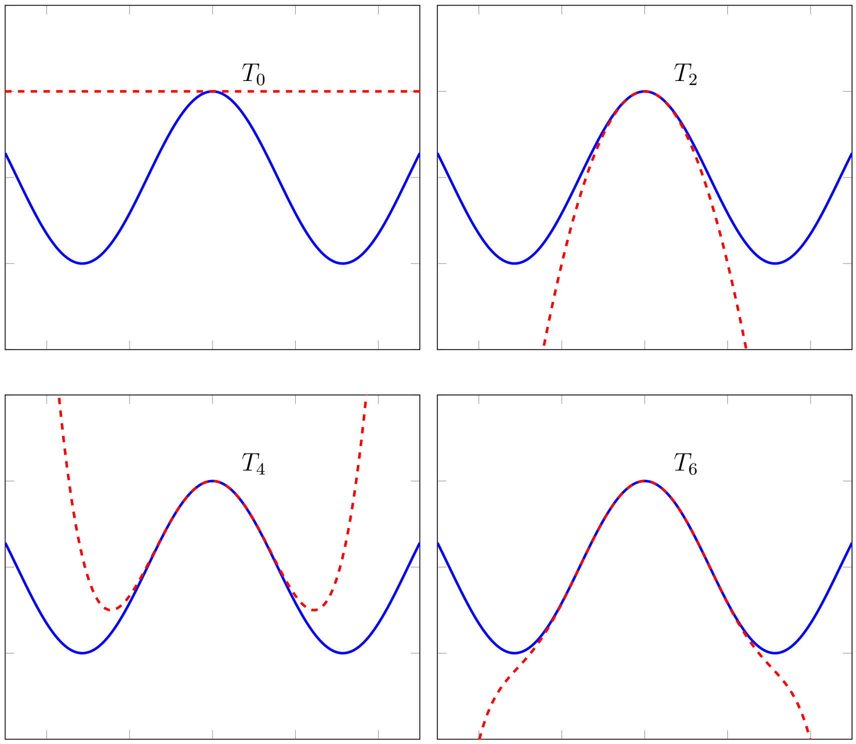
$$f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{4!}f^{(4)}(a)(x-a)^4$$

   is a polynomial whose value, and whose derivatives up to the fourth derivative, all agree with those of f at $x = a$. This polynomial is called the fourth degree Taylor polynomial of f.

   (Note: We of course saw how to compute the Taylor polynomial for other degrees, where the $(x-a)^n$ term has coefficient given by $\frac{1}{n!}f^{(n)}(a)$.)

[1] Remember that some functions don't have derivatives. For example, $f(x) = |x|$ isn't differentiable at $x = 0$.

Example 36.1.1. Let $f(x) = \cos(x)$ and set $a = 0$.



36.2 Estimating the error

So the graphs *look* similar near a, so we have graphical evidence that Taylor polynomials approximate functions well, and that for each degree we go up, the approximations look better.

But if we want to actually know that $T(0.1)$, for example, is very close to $\cos(0.1)$, how would we do that?

Goal. Given a function f, a point a, a degree d, and a "test point x," we want to answer the following question: how close is the number $T_d(x)$ (given by the Taylor polynomial) to the number $f(x)$ (the actual value of the function)?

The form of our answer. We are going to find our answer in the following form:

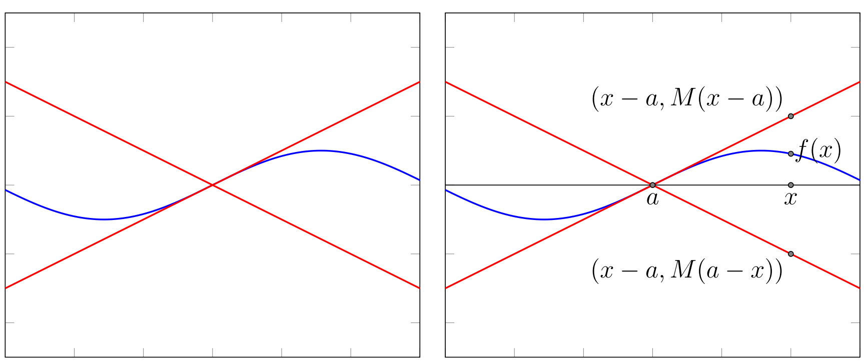
$$|f(x) - T_d(x)| \leq \text{some number we know.}$$

That is, we are going to take the number $T_d(x)$ that the Taylor polynomial tells us, and we are going to be able to compare it[2] to $f(x)$, and we are going to be able to conclude that the difference between these two numbers has absolute value *less* than some other number we know.

Example 36.2.1. Even in the degree 0 case, we'll get an intuitive answer. Suppose that the slope of f is always between $-M$ and M. Then for any x, we'll see that

$$|f(x) - T_0(x)| \leq M(x - a).$$

To interpret this, take a look at the following picture:



In blue I've drawn the graph of a function f, and in red I've drawn two lines of slopes M and $-M$, in red. Because f always has slope between M and $-M$ (meaning I

[2] Even if we don't know the exact value of $f(x)$! How cool is that?

know how f changes), I can conclude that the graph of f has to lie between the two lines I've drawn.

Moreover, in the second picture, the value of $T_0(x)$ is just the height of the horizontal black line, so the quantity $|f(x) - T_0(x)|$ is just measuring how far away the graph of f is from this horizontal black line. As you can see, this distance is sandwiched by the distance between the horizontal black line and the red slanted lines. On the other hand, the distance between the horizontal black line and the red slanted lines at x is easy to calculate—it's $M|x - a|$.

More generally, we have the following theorem:

Theorem 36.2.2 (Error bounds for Taylor polynomials)**.** Suppose you know that in the interval between x and a, the $(n + 1)$st derivative of f always satisfies

$$|f^{(n+1)}| \leq M_{n+1}.$$

(That is, the graph of $f^{(n+1)}$ is always sandwiched between the horizontal lines having height $\pm M_{n+1}$.) Then

$$|f(x) - T_n(x)| \leq M_{n+1}|(x - a)^{n+1}|. \tag{36.1}$$

I want to emphasize that when you solve problems in real life, you know $T_n(x)$, and $(x - a)$ exactly, and you can often find M_{n+1}. Thus, even if this inequality does not tell you what $f(x)$ is, it tells you that you are "at least this close" to $f(x)$ when you compute $T_n(x)$. Here, the word "this" refers to the quantity on the righthand side of the inequality (36.1).

Let's see this theorem in action.

Example 36.2.3. Let's see how good a degree one Taylor polynomial at $a = 0$ approximates sin(0.1). That is, $f = \sin(x)$ in this example.

The formula tells us

$$|\sin(0.1) - T_1(0.1)| \leq \frac{M_1}{1!}|(0.1 - 0)^1|. \tag{36.2}$$

The name of the game now is to try to find M_1. Remember, M_1 is some number that guarantees that—between x and a—the derivative of f always has absolute value smaller than M_1. Well, the derivative of sin is cos, and the value of cos is always between -1 and 1. So we know that $|f'(x)|$ will always have a value with absolute value less than or equal to 1. This means we can take M_1 to equal 1. Thus we find

$$
\begin{aligned}
|\sin(0.1) - T_1(0.1)| &\leq \frac{M_1}{1!}|(0.1-0)^1| && (36.3)\\
&= \frac{1}{1!}|(0.1-0)^1| && (36.4)\\
&= |(0.1-0)| && (36.5)\\
&= 0.1 && (36.6)
\end{aligned}
$$

Let's interpret this final inequality:

$$|\sin(0.1) - T_1(0.1)| \leq 0.1. \tag{36.7}$$

What this inequality says is that—even if I don't know what $\sin(0.1)$ is—if I plug in $T_1(0.1)$, then the resulting number will be *within* 0.1 of $\sin(0.1)$.

Shall we give it a shot? We know that

- $\sin(0) = 0$, and
- $\sin(x)' = \cos(x)$, so $f'(a) = \cos(a) = \cos(0) = 1$.

So the degree 1 Taylor polynomial centered at $a = 0$ is given by

$$T_1(x) = f(a) + f'(a)(x-a) = 0 + (x-0) = x.$$

So we see that $T_1(0.1) = 0.1$. And the inequality (36.7) tells us that $\sin(0.1)$ must be within 0.1 of 0.1. So $\sin(0.1)$ is between 0 and 0.2. Is that useful to you? Perhaps. At least you know $\sin(0.1)$ cannot be, for example, 0.20001.

Example 36.2.4. Let's do better. What we if try to approximate $\sin(0.1)$ using a *degree three* Taylor polynomial?

Then the error bound formula tells us

$$|sin(0.1) - T_3(0.1)| \leq \frac{M_4}{4!}(0.1-0)^4.$$

The fourth derivative of sin is sin again, so we can choose $M_4 = 1$. Then the righthand side of the inequality becomes

$$
\begin{aligned}
\frac{M_4}{4!}(0.1-0)^4 &= \frac{1}{4!}(0.1-0)^4 && (36.8)\\
&= \frac{1}{4!}(0.1)^4 && (36.9)\\
&= \frac{1}{4!}(10)^{-4} && (36.10)\\
&= \frac{1}{4! \times 10^4}. && (36.11)
\end{aligned}
$$

At this point, things are looking good! 10^4 is a large number, and we are dividing 1 by this large number. In fact, a calculator shows

$$\frac{1}{4! \times 10^4} < 0.00000416668 \tag{36.12}$$

so whatever you compute $T_3(0.1)$ to be, it will be within 0.00000416668 of $\sin(0.1)$. That means you can compute $\sin(0.1)$ to at least four decimal places just by plugging in 0.1 to your polynomial $T_3(x)$.

Do you want to give it a shot?

$$T_3(0.1) = 0 + 1(0.1 - 0) + 0 - \frac{1}{6}(0.1)^3 \approx 0.09983333333.$$

On the other hand, a calculator tells us

$$\sin(0.1) \approx 0.09983341665.$$

Our T_3 is looking pretty good! We even nail down the first *six* decimal places, though we were a priori guaranteed only that the first four would be correct based on (36.12).

I want to emphasize that *before Taylor polynomials*, you didn't have any tools to compute $\sin(0.1)$ by hand. All you had was a calculator, whose operations were mysterious. Now, you can compute $\sin(0.1)$—or any other value of sin—by using Taylor polynomials. And, if you choose a high enough degree polynomial, you can *certifiably*[3] compute sin to more decimal places than your calculator!

[3] That is, provably.

36.3 Why is the error bound formula true?

So why does the inequality

$$|f(x) - T_n(x)| \leq \frac{M_{n+1}}{(n+1)!}(x-a)^{(n+1)}$$

in Theorem 36.2.2 hold true?

Let's see it first when $n = 0$. In this case, we begin with the known assumption (this is how we chose M_1):

$$|f'(x)| \leq M_1 \qquad \text{when } x \text{ is close enough to } a. \tag{36.13}$$

So let's integrate both sides, assuming that $x > a$:

$$\int_a^x |f'(x)|\, dx \leq \int_a^x M_1\, dx. \tag{36.14}$$

Note that this inequality follows from (36.13) because if a function on the left always has lesser value than the function on the right, then the Riemann sum on the left will always be less than the corresponding Riemann sum on the right. So in the limit as we take more and more rectangles, the integrals will obey the same inequality.

Now we use the inequality that you studied for today:

$$|\int_a^x f'(x)\, dx| \leq \int_a^x |f'(x)|\, dx. \tag{36.15}$$

Putting together (36.14) and (36.15), we find

$$|\int_a^x f'(x)\, dx| \leq M_1(x-a). \tag{36.16}$$

Again when $x > a$, the righthand side is equal to $|x - a|$.

What if $x < a$? Then we would conclude from (36.13) that

$$\int_x^a |f'(x)|\, dx \leq \int_x^a M_1\, dx. \tag{36.17}$$

This is by the same reasoning as we saw in (36.14); but here, we are being careful that, in the bounds of integration, the number at the bottom of the integral symbol is indeed smaller than the number at the top of the integral symbol. From (36.17) and (36.15), we conclude that

$$\begin{aligned}
|f(x) - f(a)| &= |\int_a^x f'(x)\,dx| && (36.18)\\
&= |-\int_x^a f'(x)\,dx| && (36.19)\\
&= |\int_x^a f'(x)\,dx| && (36.20)\\
&\le \int_x^a M_1\,dx && (36.21)\\
&= M_1(a-x) && (36.22)\\
&= M_1|x-a| && (36.23)
\end{aligned}$$

where in the very last equality, we have used that $x < a$.

Thus, looking at both (36.16) (36.23), and noting that $T_0(x) = f(a)$ (this is special to degree 0), we can say in general that

$$|f(x) - T_0(x)| \le M_1|x-a|. \qquad (36.24)$$

This indeed agrees with Theorem 36.2.2.

The **summary of this degree zero proof** is: Just integrate the inequality $|f'| \le M_1$.

How do we do the proof for other degrees? We will follow the same strategy: **Just integrate the inequality $|f^{(n+1)}| \le M_{n+1}$ over and over.**

Let's see this in action for $n = 1$. We begin by assuming there is some real number $M_{1+1} = M_2$ so that

$$|f^{(2)}| \le M_2.$$

Let's integrate both sides from a to x to find

$$\int_a^x |f^{(2)}|\,dx \le \int_a^x M_2\,dx.$$

Applying (36.15) to the left, and integrating the righthand side, we find

$$|\int_a^x f^{(2)}\,dx| \le M_2(x-a). \qquad (36.25)$$

But the lefthand integral is, by the fundamental theorem of calculus,

$$\int_a^x f^{(2)}\,dx = f'(x) - f'(a).$$

So we find from (36.25) that

$$|f'(x) - f'(a)| \leq M_2(x - a). \tag{36.26}$$

Here is where we "integrate over and over." Let's again integrate this inequality! Then

$$\int_a^x |f'(x) - f'(a)|\, dx \leq \int_a^x M_2(x - a)dx.$$

Applying (36.15) to the left, and integrating the righthand side (sounding familiar?[4]), we find

$$|\int_a^x f'(x) - f'(a)\, dx| \leq \frac{M_2}{2}(x - a)^2. \tag{36.27}$$

Now let's integrate the lefthand side (inside the absolute value):

$$\begin{aligned}
\int_a^x f'(x) - f'(a)\, dx &= f(x) - f(a) - \left(\int_a^x f'(a)\, dx\right) && (36.28)\\
&= f(x) - f(a) - (f'(a)x - f'(a)a) && (36.29)\\
&= f(x) - f(a) - (f'(a)(x - a)) && (36.30)\\
&= f(x) - f(a) - f'(a)(x - a) && (36.31)\\
&= f(x) - (f(a) + f'(a)(x - a)) && (36.32)\\
&= f(x) - T_1(x). && (36.33)
\end{aligned}$$

Combining (36.27) with (36.33), we conclude

$$|f(x) - T_1(x)| \leq \frac{M_2}{2}(x - a)^2.$$

This is indeed the inequality in Theorem 36.2.2 for degree 1!

The pattern for proving the inequality in Theorem 36.2.2 for higher degrees is identical. You integrate over and over again, beginning with the knowledge that the $(n+1)$st derivative of f is bounded by some number M_{n+1}.

[4]See right before (36.25).

36.4 Preparation for next time

Let $f(x) = e^x$.

(a) Write out the degree 4 Taylor polynomial $T_4(x)$ for f, centered at $a = 0$.

(b) Compute $T_5(1)$. What famous number is $T_5(1)$ supposed to approximate?

(c) Somebody tells you that the sixth derivative of f is always between -3 and 3 so long as you evaluate the sixth derivative on the interval from 0 to 1. Based on this, how *close* can you guarantee $T_5(1)$ is to $f(1)$? (This is also a hint to the "famous number" question above.)

(d) Using a calculator (or by hand), write out the first 4 digits of $T_5(1)$.

(e) How does this compare to the "famous number"?

36.4.1 Plus One

There is no plus one problem for this last week, as there is realistically no time for people to come to office hours after these are graded.