Lecture 37

Differentiable functions are continuous

Today, we are going to "review" some notions of continuity and differentiability. At the same time, we will learn the following fact:

Theorem 37.0.1. Let f be a function, and a a number. If f is differentiable at a, then f is continuous at a.

(This is a logical statement that is very straightforward in form. For example, the statement "Let R be a polygon. If R is a square, then it is a rectangle" has a logically identical form.)

To appreciate the theorem, let's review what the key words in the theorem mean.

First, let's recall:

Definition 37.0.2. The *derivative of f at a* is the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, we say that the derivative exists, and we also say that f is differentiable at a.

If the limit does not exist, we say that the derivative does not exist at a, and we also say that f is not differentiable at a.

When the derivative at a does exist, we write it as

$$f'(a).$$

Finally, we write

$$f', \qquad \text{or} \qquad f'(x)$$

for the new function we obtain by declaring that f' sends any number x to the derivative of f at x (if it exists). f' is not defined at x if f' is not differentiable at x.

37.1 Example of using definition of derivative

Example 37.1.1. You "know" that the derivative of $f(x) = x^2 + 3$ at $a = 5$ is given as follows:

$$f'(x) = 2x \text{ so } f'(a) = f'(5) = 2(5) = 10.$$

You used the power rule to compute this. However, note that in the work just shown, you did *not* use the definition of the derivative!

So let's compute it using the definition of derivative. The derivative of f at $a = 5$ is given by:

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} &(37.1)\\
&= \lim_{h\to 0} \frac{(5+h)^2 + 3 - (5^2+3)}{h} &(37.2)\\
&= \lim_{h\to 0} \frac{5^2 + 10h + h^2 + 3 - 5^2 - 3}{h} &(37.3)\\
&= \lim_{h\to 0} \frac{10h + h^2}{h} &(37.4)\\
&= \lim_{h\to 0} (10 + h) &(37.5)\\
&= 10 + \lim_{h\to 0} h &(37.6)\\
&= 10 + 0 &(37.7)\\
&= 10. &(37.8)
\end{aligned}$$

What things like the power rule, Leibniz rule, chain rule, et cetera do is give you a *shortcut*. Just like you have memorized that $9 \times 9 = 81$ (and you do not add 9 to itself nine times), the derivative rules allow you to skip the limit definition of derivative and go straight to computing an answer.

37.2 Another example of using definition of derivative

Example 37.2.1. Likewise, you *proved* that the derivative of sin at 0 is given by 1 using the limit definition:

$$\begin{aligned}
\sin'(0) &= \lim_{h\to 0} \frac{\sin(0+h)-\sin(0)}{h} && (37.9)\\
&= \lim_{h\to 0} \frac{\sin(0)\cos(0)+\sin(h)\cos(0)-\sin(0)}{h} && (37.10)\\
&= \lim_{h\to 0} \frac{0\cdot\cos(h)+\sin(h)\cdot 1-0}{h} && (37.11)\\
&= \lim_{h\to 0} \frac{\sin(h)}{h} && (37.12)
\end{aligned}$$

and we showed that this limit is 1 in class, using the squeeze theorem. If you want to construct the formula for $\sin(x)$ in general, we can compute:

$$\begin{aligned}
\sin'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} && (37.13)\\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} && (37.14)\\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\sin(h)\cos(x)}{h} && (37.15)\\
&= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0}\frac{\sin(h)\cos(x)}{h} && (37.16)\\
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} && (37.17)\\
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x). && (37.18)
\end{aligned}$$

Now I claim that $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$. To see this, let's make the substitution $\theta = h/2$. Then $\cos(h) = \cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2(\theta)$. That is,

$$\cos(h) = 1 - 2\sin^2(h/2).$$

So we find

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = \lim_{h\to 0} \frac{1-2\sin^2(h/2)-1)}{h} \tag{37.19}$$
$$= \lim_{h\to 0} \frac{2\sin^2(h/2)}{h} \tag{37.20}$$
$$= \lim_{h\to 0} \frac{2\sin^2(h/2)}{2(h/2)} \tag{37.21}$$
$$= \lim_{h\to 0} \frac{\sin^2(h/2)}{(h/2)} \tag{37.22}$$
$$= \lim_{h\to 0} \frac{\sin(h/2)}{(h/2)} \sin(h/2) \tag{37.23}$$
$$= \lim_{h\to 0} \frac{\sin(h/2)}{(h/2)} \lim_{h\to 0} \sin(h/2) \tag{37.24}$$
$$= 1 \cdot 0 \tag{37.25}$$
$$= 0. \tag{37.26}$$

Remark 37.2.2. As you can see, being able to compute the derivative of sin really relies on being able to compute some difficult limits! When you memorize the formula $\frac{d}{dx}\sin(x) = \cos(x)$, you have stood on the shoulders of giants (i.e., skipped the labor of computing these limits) so that you can see farther.

37.3 Continuity

Let's get back to our main theorem, Theorem 37.0.1. It says that if f is differentiable at a, then f is continuous at a. We have just reviewed that being "differentiable at a" means that a certain limit (the limit of the difference quotient) exists.

So let's review what it means for f to be continuous.

Definition 37.3.1. We say that f is continuous at a if

1. f is defined at a

2. $\lim_{x \to a} f(x)$ exists, and

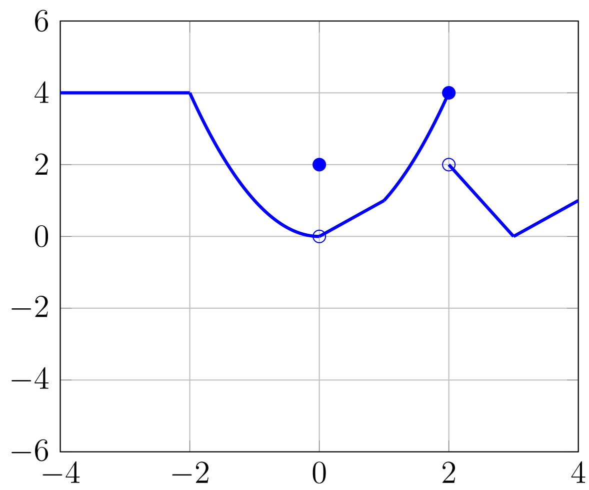
3. $f(a) = \lim_{x \to a} f(x)$.

The most important part of the definition of continuity is the last equation. Note that if you even want to write an equality like $f(a) = \lim_{x \to a} f(x)$, then you need to know that you have numbers on both sides of the equality! That's why you demand that f is defined at a (so that $f(a)$ makes sense) and that the limit exists (so that $\lim_{x \to a} f(x)$ is a number).

Remark 37.3.2. Just as with the definition of "derivative," the definition of "continuity" involves the notion of limits. So you need to know how to determine whether a limit exists, and how to compute it, to test for continuity.

In this class, you have learned some shortcuts about testing for continuity—just like you learned derivative rules. For example, if you are given the graph of a function f, you know that f is continuous so long as there are no "jumps" in f. We described this as "if you can draw the graph of f without lifting your pencil, then f is continuous."

We also saw how to think about graphs of f as follows:



This picture means that f takes on the value $f(0) = 2$, and $f(2) = 4$, for example. (The *white dots* mean that f does *not* take on the white-dot value, while the black dots signify the values that f *does take*.)

37.4 Differentiability implies continuity

So, let's see why f being differentiable at a ensures that f is continuous at a. This means we need to check the three conditions in the definition of continuity:

1. If f is differentiable at a, is f defined at a?

The answer here is yes. In fact, that f is defined at a is a requirement for being able to define differentiability. After all, when computing the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

we have to know that a number called $f(a)$ is given (in order to compute the numerator).

2. If f is differentiable at a, does $\lim_{x\to a} f(x)$ exist? This is the hard part.

First, we know that f is differentaible at a. Let's call $f'(a)$ a concrete symbol, called k, just to save space. Then

$$k = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Let's manipulate this equation cleverly:

$$\begin{aligned}
0 &= \left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right) - k && (37.27)\\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} - \lim_{h\to 0} \frac{kh}{h} && (37.28)\\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)-kh}{h} && (37.29)\\
0 \cdot \lim_{h\to 0} h &= \lim_{h\to 0} \frac{f(a+h)-f(a)-kh}{h} \cdot \left(\lim_{h\to 0} h\right) && (37.30)\\
0 &= \lim_{h\to 0} \left(\frac{f(a+h)-f(a)-kh}{h} \cdot h\right) && (37.31)\\
0 &= \lim_{h\to 0} \left(f(a+h)-f(a)-kh\right) && (37.32)\\
& && (37.33)
\end{aligned}$$

Let's see how we did these steps. We began by subtracting the number k from both sides, obtaining the first line. Then we used the fact that this number k equals a limit, $\lim_{h\to 0} \frac{kh}{h}$, to obtain (37.28).

To obtain (37.29), we used the *addition law* for limits. Remember that this says that if the limits of two expressions g and h exist, then the limit of $g + h$ exists. Moreover, $\lim g + h = \lim g + \lim h$.

To obtain the next line (37.30), we multiplied both sides by a number called 0, but expressed as $\lim_{h\to 0} h$. The reason for using this expression is to use the *product law* for limits, which says $\lim g \cdot \lim h = \lim gh$, provided that the limits $\lim g$ and $\lim h$ both exist. The product law allows us to go from (37.30) to (37.31). Finally, the last line is obtained by a straightforward cancellation of h.

How does this help? Let's keep going:

$$\begin{aligned}
0 &= \lim_{h\to 0} \left(f(a+h) - f(a) - kh\right) && (37.34)\\
&= \lim_{h\to 0} \left(f(a+h) - f(a) - kh\right) + \lim_{h\to 0} \left(f(a) + kh\right) - \lim_{h\to 0} \left(f(a) + kh\right) && (37.35)\\
&= \lim_{h\to 0} \left(f(a+h) - f(a) - kh + (f(a) + kh)\right) - \lim_{h\to 0} \left(f(a) + kh\right) && (37.36)\\
&= \lim_{h\to 0} \left(f(a+h)\right) - \lim_{h\to 0} \left(f(a) + kh\right) && (37.37)\\
& && (37.38)
\end{aligned}$$

First, let's note that we know there is a number called $\lim_{h\to 0}(f(a) + kh)$; this limit is straightforwardly computed to equal $f(a)$. But we're going to keep this number written as a limit. Then equation (37.35) is obtained from the first line by adding and subtracting this expression. (Note that if we add, and then subtract this equation, we are simply adding a term equal to 0! So this does not change the equality.)

We arrive at (37.36) by using the addition law for limits again. Note it is important that we know every limit in the previous step existed; that is what guarantees we can use the addition law and conclude that the limit $\lim_{h\to 0} \left(f(a+h) - f(a) - kh + (f(a) + kh)\right)$ exists.

Then (37.37) is deduced by simply canceling some terms inside the limit.

Importantly, note that we see in (37.37) that the limit $\lim_{x\to a} f(x)$ exists. Making the substitution $x = a + h$, the limit $x \to a$ is computed precisely as $h \to 0$. That is,

$$\lim_{h\to 0} f(a+h) = \lim_{x\to a} f(x). \tag{37.39}$$

This is because the function $j(h) = a+h$ is continuous, so the limit $\lim_{h\to 0} f(j(h)) = \lim_{x\to j(h)} f(x)$.

3. If f is differentiable at a, does $\lim_{x\to a} f(x)$ equal $f(a)$? Let us continue:

$$
\begin{aligned}
&= \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) - \lim_{h\to 0} kh && (37.40)\\
&= \left(\lim_{h\to 0} f(a+h)\right) - f(a) - \lim_{h\to 0} kh && (37.41)\\
&= \left(\lim_{h\to 0} f(a+h)\right) - f(a) - k \lim_{h\to 0} h && (37.42)\\
&= \left(\lim_{h\to 0} f(a+h)\right) - f(a) - k \cdot 0 && (37.43)\\
&= \left(\lim_{h\to 0} f(a+h)\right) - f(a) - 0 && (37.44)\\
&= \left(\lim_{h\to 0} f(a+h)\right) - f(a) && (37.45)\\
f(a) &= \lim_{h\to 0} f(a+h). && (37.46)
\end{aligned}
$$

(37.40) is obtained by applying the addition law to (37.37).

The next few lines use the fact that $f(a)$ and k are concrete numbers—that is, constants. This gets us to (37.45).

The passage from (37.45) to (37.46) is just adding $f(a)$ to both sides. Finally, using (37.39) again, we conclude that

$$f(a) = \lim_{x\to a} f(x).$$

That was a long, long path, but we proved our theorem (Theorem 37.0.1): If f is differentiable at a, it is continuous at a. We proved this by verifying all three conditions in the definition of continuity.

37.5 Preparation for next time

Let $f(x) = 3x + 5$. Let $a = 2$.

(a) Let $L = 11$ and $\epsilon = 0.1$. Exhibit a number δ guaranteeing the following: Whenever $|x - a| < \delta$, you know $|f(x) - L| < \epsilon$.

(b) If $L = 11$, is L equal to the limit $\lim_{x \to a} f(x)$?

(c) Now change L to equal 10, and choose $\delta = 0.1$. (This time, we are fixing δ, not ϵ.) Exhibit a number ϵ guaranteeing the following: Whenever $|x - a| < \delta$, you know $|f(x) - L| < \epsilon$.

(d) If $L = 10$, is L equal to the limit $\lim_{x \to a} f(x)$?

37.5.1 Plus One

There is no plus one problem for this last week, as there is realistically no time for people to come to office hours after these are graded.