Given two matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

ROAN-WATSON metric:

$$\sqrt{(a-a')^2+(b-b')^2} + \sqrt{(c-c')^2+(d-d')^2} + \sqrt{(a-a')^2+(c-c')^2} + \sqrt{(b-b')^2+(d-d')^2} = d_{RW}$$

Pohlman attempt: $|(a-a')+(b-b')+(c-c')+(d-d')|$
this fails PROPERTY ①

PROPERTY ①
If $A=A'$, then $d(A,A')=0$
If $d(A,A')=0$, then $A=A'$.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

indeed $d_{RW}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = 0.$
$\quad \parallel$
$d_{RW}(A, A')$.

Pohlman metric: $\sqrt{(a-a')^2+(b-b')^2+(c-c')^2+(d-d')^2} = d_P(A,A')$

(this is the metric induced by
- the bijection $\theta: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$
- the standard metric on $\mathbb{R}^4$.

$|a-a'|+|b-b'|+|c-c'|+|d-d'| = d_G(A,A')$ GARZA metric.

$\max\{|a-a'|, |b-b'|, |c-c'|, |d-d'|\} = d_{vdG}(A,A')$ van de Geijn

Thm $\forall\, n \in \mathbb{Z}, n > 0,$

$$\left(|a-a'|^n + |b-b'|^n + |c-c'|^n + |d-d'|^n \right)^{1/n}$$ is a metric.

Natural q: what if $n \to \infty$? **Thm** $\lim_{n\to\infty} \left(|a-a'|^n + \dots + |d-d'|^n \right)^{1/n} = \max\{|a-a'|, \dots, |d-d'|\}$ (ℓ^∞ metric)

$$d_T(A,A') = \begin{cases} 0 & \text{if } A = A' \\ 1 & \text{otherwise} \end{cases}$$ Tucci metric (discrete metric)

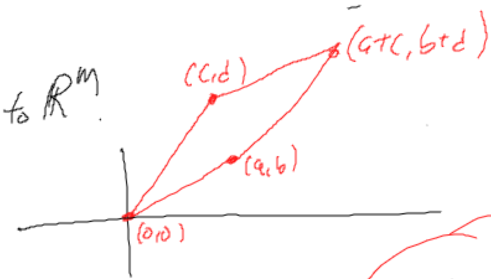
Obs All these $\underbrace{\text{metrics involve}}$ (except d_{Tucci}) a, a', b, b', etc (the entries).

Q: Is there a way to express a metric (some metric) w/o entries?

- What is L^2? ℓ^2?
- What's the distance btwn two parallelograms?
  - does these parallelograms depend on entries?

A matrix "is" a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

A matrix can be visualized as a parallelogram:



Given a 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have a fxn

$(x_1, x_2) \in \mathbb{R}^2 \to \mathbb{R}^2$,

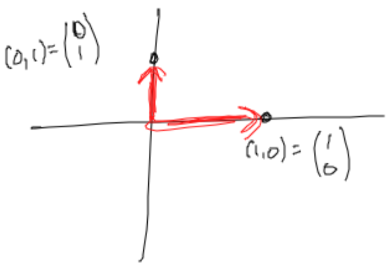
$$m\left\{ \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix}}_{n} \right. \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

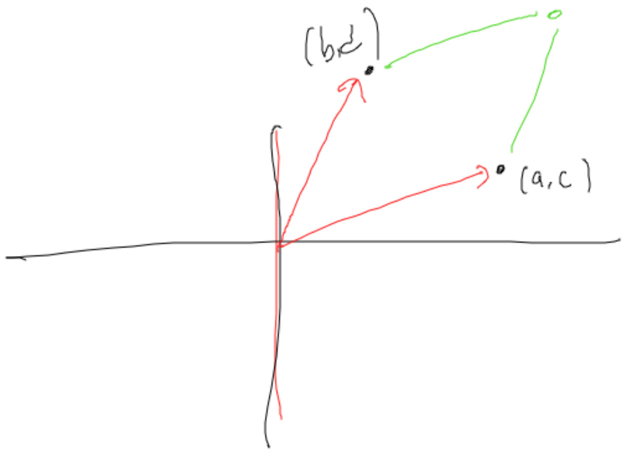
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

$(ax_1 + bx_2, cx_1 + dx_2)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a\cdot 1 + b\cdot 0 \\ c\cdot 1 + d\cdot 0 \end{pmatrix} = \begin{pmatrix} a+0 \\ c+0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$





So we can think of A as a function $\mathbb{R}^2 \to \mathbb{R}^2$

Then: Is there a notion of distance between two functions?

Topology 4330

$\{f : [0,1] \to \mathbb{R}\}$

WELSH

$\{f : \mathbb{R}^n \to \mathbb{R}\}$

Fix a fxn $f : \mathbb{R} \to \mathbb{R}$.

f is called L^2 if

$$\int_{\mathbb{R}} |f|^2 \, dx < \infty.$$

our classmates have seen distance b/twn fxns like

and this involved integration.

$L^2(\mathbb{R}) := \{$fxns f that are $L^2\}$.

$$d_{L^2}(f,g) = \left(\int |f-g|^2 \, dx \right)^{1/2}$$

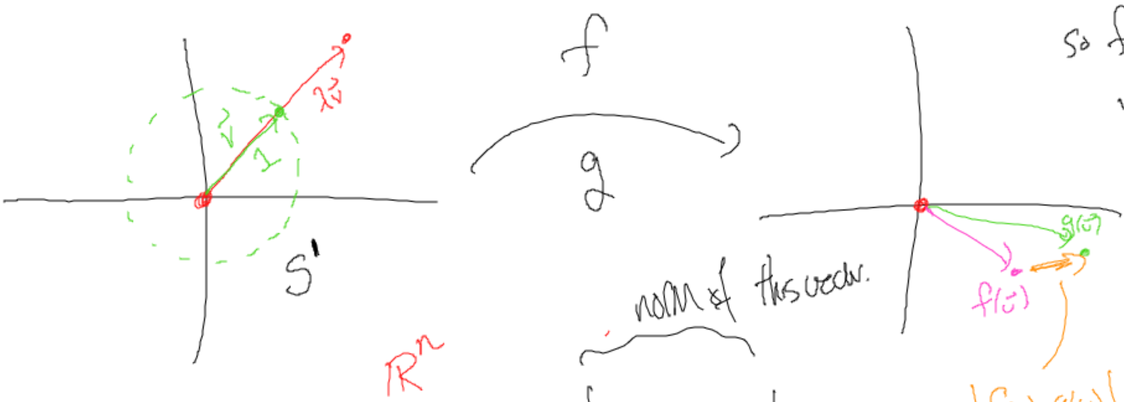
$x \in \mathbb{R}$

$f(x),\ g(x)$

Distances btwn <u>linear</u> transf. from $\mathbb{R}^m$ to $\mathbb{R}^n$

$f(\lambda\vec{v}) = \lambda f(\vec{v})$

So f is determined by what it does on unit vectors



ie, by what it does on the unit circle, S^1.

$|f(v)-g(v)|$ = length.

So, $\forall\ \vec{v} \in S^1$, consider $\overbrace{|f(\vec{v})-g(\vec{v})|}^{\text{norm of this vector}}$, then

$$\max_{\vec{v} \in S^{n-1}} \{ |f(\vec{v})-g(\vec{v})| \} = d_{\text{operator norm}}(f, g).$$

Q: We have 5 different metrics on $M_{2\times 2}(\mathbb{R})$. How many topologies do these metrics lead to?