$(AB)^T = B^T A^T$

Orthogonal matrices:

An $n\times n$ matrix A is called orthogonal iff $A^T A = I$

$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\Rightarrow \boxed{\langle u, v\rangle = \sum_{i=1}^{n} x_i y_i = u^T v}$

$(x_1 \cdots x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (x_1 y_1 + \cdots + x_n y_n)$

Ex $\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} = A$

$$\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} = \begin{pmatrix} \frac{9}{25}+\frac{16}{25} & -\frac{12}{25}+\frac{12}{25} \\ -\frac{12}{25}+\frac{12}{25} & \frac{16}{25}+\frac{9}{25} \end{pmatrix} = \begin{pmatrix} \frac{25}{25} & 0 \\ 0 & \frac{25}{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Prop. Let $u, v \in \mathbb{R}^n$. If A is orthogonal,

$\langle u, v\rangle = \langle Au, Av\rangle$

Pf. $\langle u, v\rangle = u^T v = u^T I v = u^T (A^T A) v = (u^T A^T)(Av) = (Au)^T (Av) = \langle Au, Av\rangle$

Remark

A Preserving $\langle -,- \rangle \implies$ A Preserving $\measuredangle$s and distances.

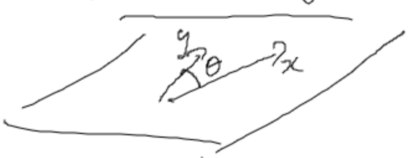
$$\sqrt{\langle y-x, y-x\rangle} = \sqrt{(y_1-x_1)(y_1-x_1)+\cdots+(y_n-x_n)(y_n-x_n)} = \sqrt{(y_1-x_1)^2+\cdots+(y_n-x_n)^2}$$

Let's recall $\text{dist}(x,y) = \sqrt{\langle y-x, y-x\rangle}$.

Then
$$\sqrt{\langle y-x, y-x\rangle} \overset{\text{b/c } A \text{ preserves } \langle , \rangle}{=} \sqrt{\langle A(y-x), A(y-x)\rangle}$$
$$\overset{\text{b/c } A \text{ linear}}{=} \sqrt{\langle Ay-Ax, Ay-Ax\rangle}$$
$$\overset{\text{by defn of distance}}{=} \text{dist}(Ay, Ax).$$

Thus $\text{dist}(x,y) = \text{dist}(Ax, Ay)$.

$$|x| = \text{dist}(x,0) = \text{dist}(Ax, A0) = \text{dist}(Ax, 0) = |Ax|$$
$\Uparrow$



Recall: $\langle x,y\rangle = |x|\,|y| \cdot \cos(\theta)$
$\langle Ax, Ay\rangle = |Ax|\,|Ay| \cos(\theta_A)$

$\uparrow$ θ = angle from x to y in the plane spanned by x, y

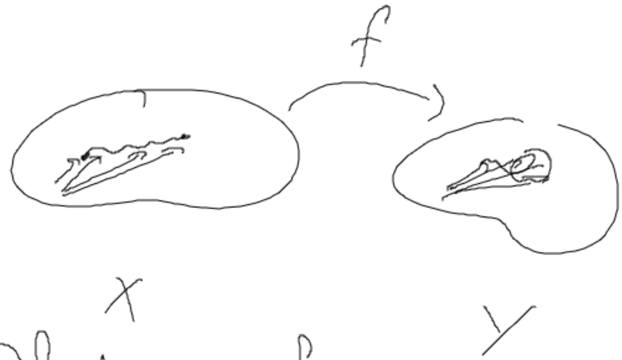
BTW • Any linear transformation

$$A: \mathbb{R}^n \to \mathbb{R}^m$$

is continuous.

Cor. Any invertible $n \times n$ matrix A defines a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$.

Pf. By BTW, A is continuous. B/c A is invertible, A is a bijection, A^{-1} is a linear transf., so A^{-1} is contin. //



Defn. A continuous fxn

$$f: X \to Y$$

is called a homeomorphism if

- f is a bijection
- f^{-1} is continuous.

$O_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$.

Notation

We let $\overbrace{O_n(\mathbb{R})}^{\text{the orthogonal group.}}$ denote the set of $n\times n$ real orthogonal matrices.

We say that an orthogonal matrix A is **special** if $\det A = 1$.

We let $\overbrace{SO_n(\mathbb{R})}^{\text{the special orthogonal group.}}$ denote the set of special orthogonal matrices.

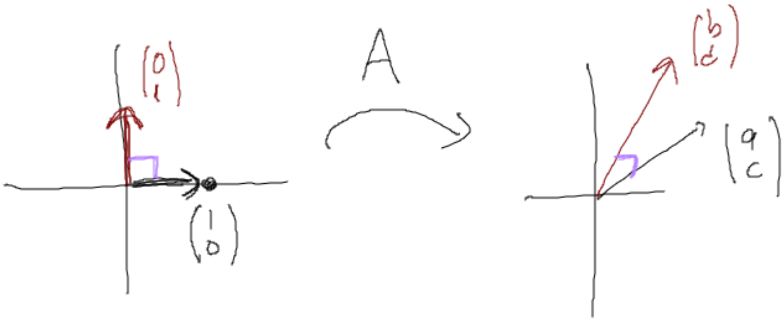
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Tucci: If A is orthogonal, the columns of A are linearly independent.

(H: For 2×2 matrices, this means neither column is a multiple of the other.)

Q: What does "$\det(A) = 1$" buy us?

HINTS: If A is orthogonal, what can you say about the columns of A?



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$(1 \quad 0) \leftarrow \cdot$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot a + 0 \cdot b \\ 1 \cdot c + 0 \cdot d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$