Compare

$S^1 \times S^1$ and S^2.

$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$

$S^1 \subset \mathbb{R}^2$

What is $S^1 \times S^1$?

$S^1 \times S^1 = \{(x, y) \mid x \in S^1, y \in S^1\}$

$x \in S^1 \subset \mathbb{R}^2$, $x = (x_1, x_2)$

$((x_1, x_2), (y_1, y_2)) \in S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$ $\quad (x_1, x_2, y_1, y_2)$

Each $\theta \in \mathbb{R}$

determines an element of S^1.

$$\theta \longmapsto (\cos(\theta), \sin(\theta))$$

$$[0,2\pi] \overset{\iota}{\hookrightarrow} \mathbb{R} \xrightarrow{\gamma} S^1$$

$\gamma \circ \iota : [0,2\pi] \to S^1$

$[0,2\pi] \subset \mathbb{R}$

$\gamma \circ \iota$

$A \subset X$

$\iota : A \to X$

$a \longmapsto a$

inclusion fxn

ι is a continuous fxn (when you give A the topology induced by X)

Ex subgp

$H \subset G$

$\iota : H \to G$

$h \longmapsto h$

ι is a gp homom.

Declare $\theta \sim \theta' \iff \gamma \circ \iota(\theta) = \gamma \circ \iota(\theta')$.

Theorem: $[0,2\pi]/\sim$ is homeom. to S^1.

NOTATION

Fix a set X and an equivalence relation $\sim$ on X.

Then $X/\sim$ is the set of equivalence classes of X.

Defn An equivalence relation on a set X is a subset $R \subset X \times X$ s.t.

" $x \sim y$ " $\Leftrightarrow$ " $(x,y) \in R$ "

(1) $\forall x \in X$, $(x,x) \in X \times X$ is an element in R.

$x \sim x$

(2) $\forall x,y,z \in X$, if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$

$x \sim y,\ y \sim z \Rightarrow x \sim z$

(3) $\forall x,y \in X$, $(x,y) \in R \Leftrightarrow (y,x) \in R$.

$x \sim y \Leftrightarrow y \sim x$

Defn An equivalence class is a subset $E \subset X$ s.t.

(i) $\phi \neq E$

(ii) If $x,y \in E$ then $x \sim y$

(iii) If $x \in E$, $y \sim x$, then $y \in E$.

$[x]$ $\Updownarrow$ $x \in [x]$

$$X/\sim = \{[x] \mid x \in X\}$$

The point of passing from X to $X/\sim$ is to

- glue points together ($x \neq y$, but $x \sim y$ $\Rightarrow [x]=[y]$)
- remove redundancies.

Note:

$$X \xrightarrow{\text{quotient}} X/\sim$$

$$x \longmapsto [x]$$

FACT If X is a topological space and $\sim$ is an equiv. rel'n, then $X/\sim$ has a topology for which $X \to X/\sim$ is continuous (and universal).

$$[0,2\pi] \subset \mathbb{R} \xrightarrow{\;p\;} S^1 \implies [0,2\pi]/\sim \;\cong S^1$$

└ p is a "covering map."

↑ homeomorphism

∀ covering maps, $p: X \to Y$, $Y \cong X/\sim$

$x \sim x' \iff p(x) = p(x')$.

$0 \sim 2\pi$ $\quad$ $\pi \sim 3\pi$

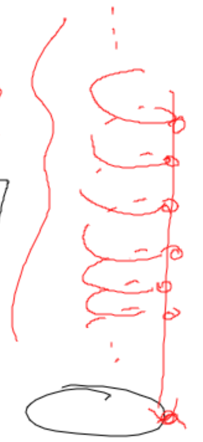
$2\pi \sim 4\pi$

$$\theta \sim \theta + 2\pi k, \; k \in \mathbb{Z}$$

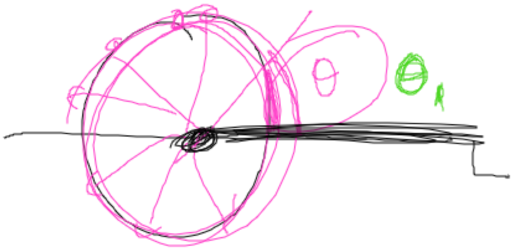
<u>Thm</u> $\mathbb{R}/\sim \;\longrightarrow S^1$ is a homeomorphism

$$[\theta] \longmapsto (\cos(\theta), \sin(\theta))$$

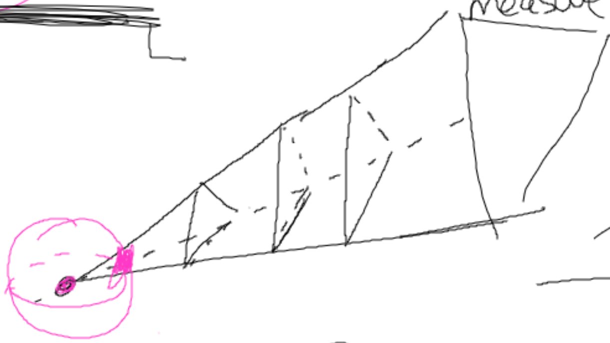
$\mathbb{R}$

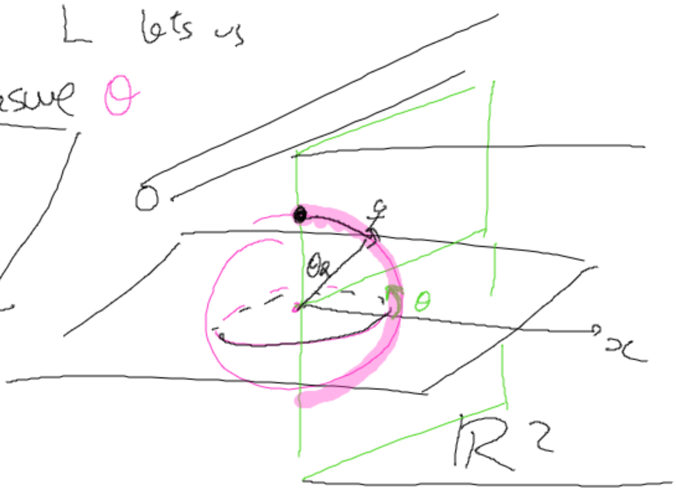


S^1



Choice of L lets us measure θ





For S^2: $\mathbb{R}^2 \longrightarrow S^2$

$(\theta_1, \theta_2) \longmapsto (\cos\theta_2, \cos\theta_1 \sin\theta_2, \sin\theta_1 \sin\theta_2)$

How do we think of $S^1 \times S^1$? Can we compare it to S^2 ?

We can think of a point in $S^1 \times S^1$ as a pair (θ_1, θ_2).

$$[0,2\pi]/\sim \xrightarrow{f} S^1 \quad \text{is a homeom.}$$
$$[\theta] \longmapsto (\cos\theta, \sin\theta)$$

So

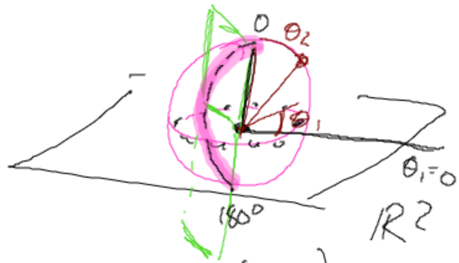
$$[0,2\pi]/\sim \times [0,2\pi]/\sim \xrightarrow{f \times f} S^1 \times S^1 \quad \text{is a homeom.}$$
$$([\theta_1], [\theta_2]) \longmapsto ((\cos\theta_1, \sin\theta_1), (\cos\theta_2, \sin\theta_2))$$

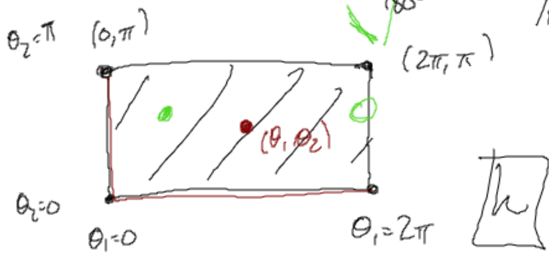
$$S^1 \times S^1 \xleftarrow{\;g\;} \mathbb{R}^2 \xrightarrow{\;h\;} S^2 \qquad \mathbb{R}^2/\sim$$

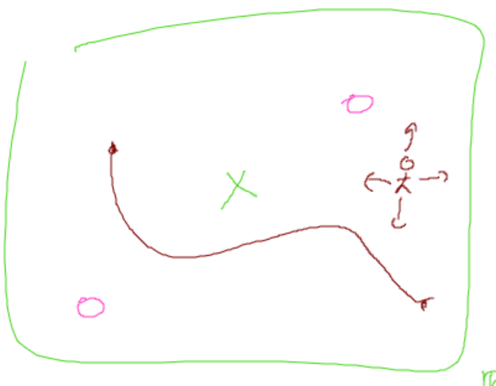
$$((\cos\theta_1, \sin\theta_1), (\cos\theta_2, \sin\theta_2)) \longleftarrow\!\!\mid (\theta_1, \theta_2) \longmapsto (\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2)$$

When do two pts in $\mathbb{R}^2$ have the same image under g? under h?

NEXT TIME understand which pts have same image under g, h







$\mathbb{R}^2 \setminus \{0\}$