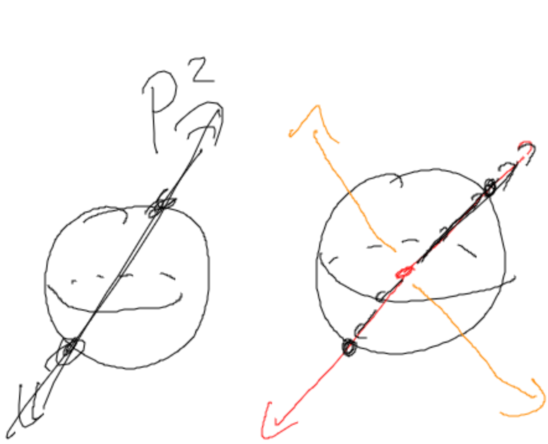
Map of Earth PacMan

$\mathrm{int}(D^2)$

$\cong$ homeo

$\mathbb{R}^2$ PLANE

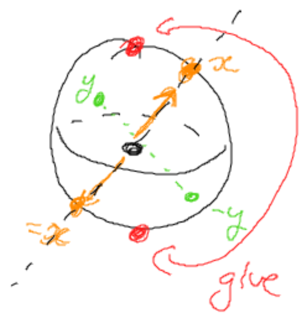
$$\underbrace{\mathrm{int}(D^2)}_{\mathbb{R}^2} \cup \mathbb{R}P^1 = \underline{\underline{\mathbb{R}P^2}}$$



How to define

- P^2
- L

to define a surjection $S^2 \to P^2$



$x = (x_1, x_2, x_3)$

$-x = (-x_1, -x_2, -x_3)$

$$P^2 = \{ L \subset \mathbb{R}^3 \mid \underline{L \ni 0}, \ \underline{L \text{ a line}} \}$$

$S^2/_{x\sim -x} \ni [x]$

Tucci: There's a 1-1 correspondence between P^2 and $S^2/\sim$.

$L \longleftrightarrow [x] = \{x, -x\}$

the line through $x, -x$ $\longleftrightarrow$ $[x] = \{x, -x\}$

$$\underline{P^2} \xleftarrow{AR} \underline{S^2/\sim}$$

$$\boxed{\begin{array}{l} P^2 \xrightarrow{PT} S^2/\sim \\ L \longmapsto L \cap S^2 \end{array}}$$

Thm $\exists$ a bijection

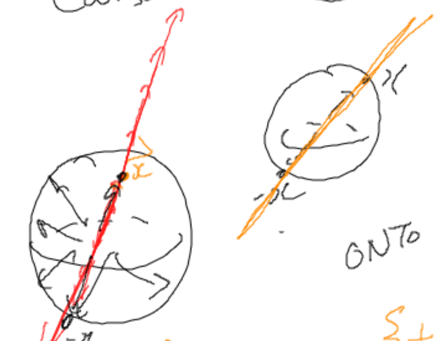
$$P^2 \rightleftarrows S^2/\sim$$

$$L \longrightarrow L \cap S^2$$

$$S^2 \twoheadrightarrow S^2/\sim \dashrightarrow P^2$$

Ex Cr:

Construct $S^2 \twoheadrightarrow P^2$ "$\twoheadrightarrow$" surjection.



ONTO b/c every line thru $O \in \mathbb{R}^3$ intersects S^2.

$\vec{x} \in S^2 \longmapsto \{t\vec{x} \mid t \in \mathbb{R}\} \in P^2$.

This is a 2:1 (two-to-one) map.

Often, we write

two pts x, y on S^n are called antipodal if $x = -y$.

$\mathbb{RP}^1$ instead of P^1

$\mathbb{RP}^2$ P^2

P stands for "projective".

$P^1 = \mathbb{RP}^1$, "the real projective line"

$P^2 = \mathbb{RP}^2$. "the real projective plane"

$\mathbb{RP}^1 = P^1 \xleftarrow{2:1} S^1$

the line thrgh O and x $\longleftarrow\!\shortmid$ x;

$\mathbb{RP}^2 = P^2 \xleftarrow{2:1} S^2$

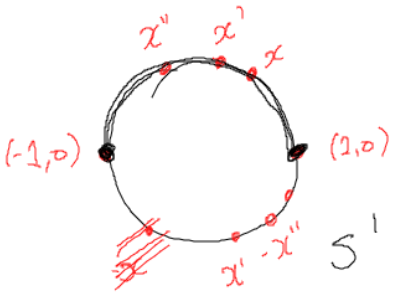
line thrgh x and O $\longleftarrow\!\shortmid$

$\mathbb{RP}^n \xleftarrow{2:1} S^n$

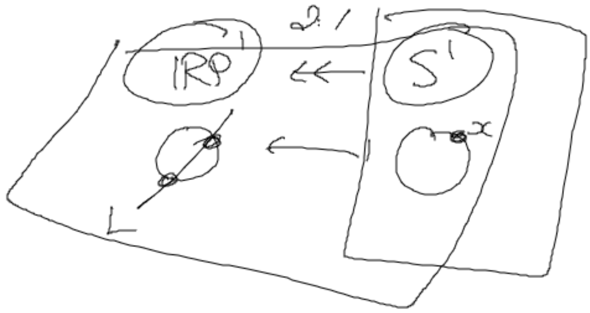
line thrgh $x, -x$. $\longleftarrow\!\shortmid$ x

$\boxed{\mathbb{RP}^1}$

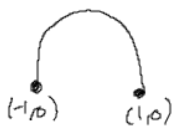
$\mathbb{RP}^1 = \{$ lines through $0 \in \mathbb{R}^2 \}$



S^1

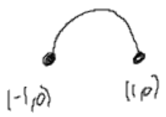


" $S^1 / x \sim -x$ looks a lot like



modulo $(1,0) \sim (-1,0)$ "

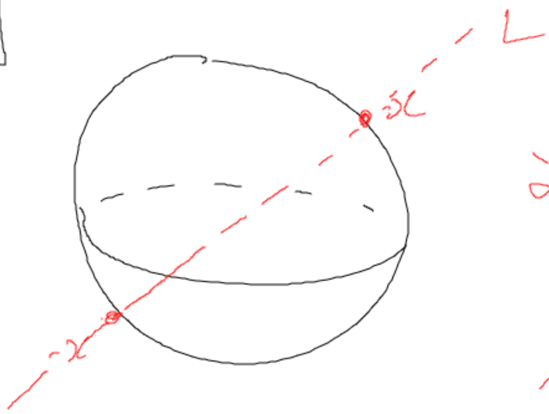
So $\mathbb{RP}^1$ looks like

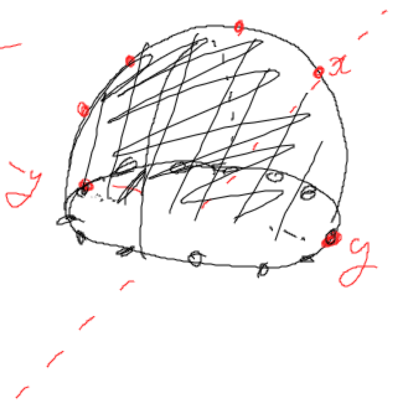


modulo $(1,0) \sim (-1,0)$.

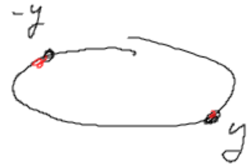
$\boxed{\text{Prop'n } \mathbb{RP}^1 \cong S^1}$

$\boxed{\mathbb{RP}^2}$





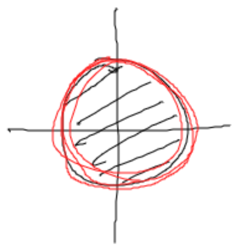
$\mathbb{RP}^2 \xleftarrow{2:1} S^2$



$\mathbb{RP}^2$ looks like northern hemisphere / $y \sim -y$ if y is on equator.

$\mathbb{RP}^2 =$ (strictly) northern hemisphere $\cup$ $\mathbb{RP}^1$.

$\cong$



$$D^2 := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \mathrm{dist}(x, 0) \le 1\}$$

interior

$$\mathrm{int}(D^2) := \{x \mid \mathrm{dist}(x, 0) < 1\}$$

Last slide: $\mathbb{R}P^2 = \mathrm{int}(D^2) \cup \mathbb{R}P^1$

Theorem There is no continuous injection of $\mathbb{RP}^2$ into $\mathbb{R}^3$.

IF f respects $\sim$

$$\begin{tikzcd} S^2 \arrow[r, hook, "f"'] \arrow[d] & \mathbb{R}^3 \\ \mathbb{RP}^2 = S^2/\sim \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

NO GO.

$x \sim -x$

$\Rightarrow f(x) = f(-x)$

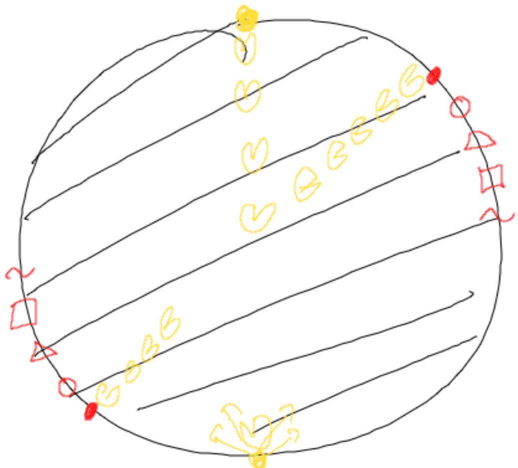
FACTS There does exist

- a continuous injection of S^2 into $\mathbb{R}^3$.

  (The inclusion fxn $\imath : S^2 \to \mathbb{R}^3$)

- a continuous inj. of 

So how to visualize?



D^2

Q: Can contract this loop to a point?!

$\mathbb{R}P^2$ PacMan

$\mathbb{R}P^2$ PacMan