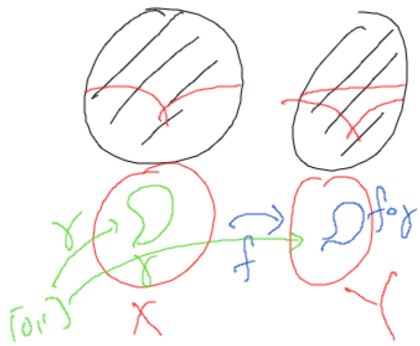
<u>Facts</u>: In the examples below, choose whatever x_0 you want.

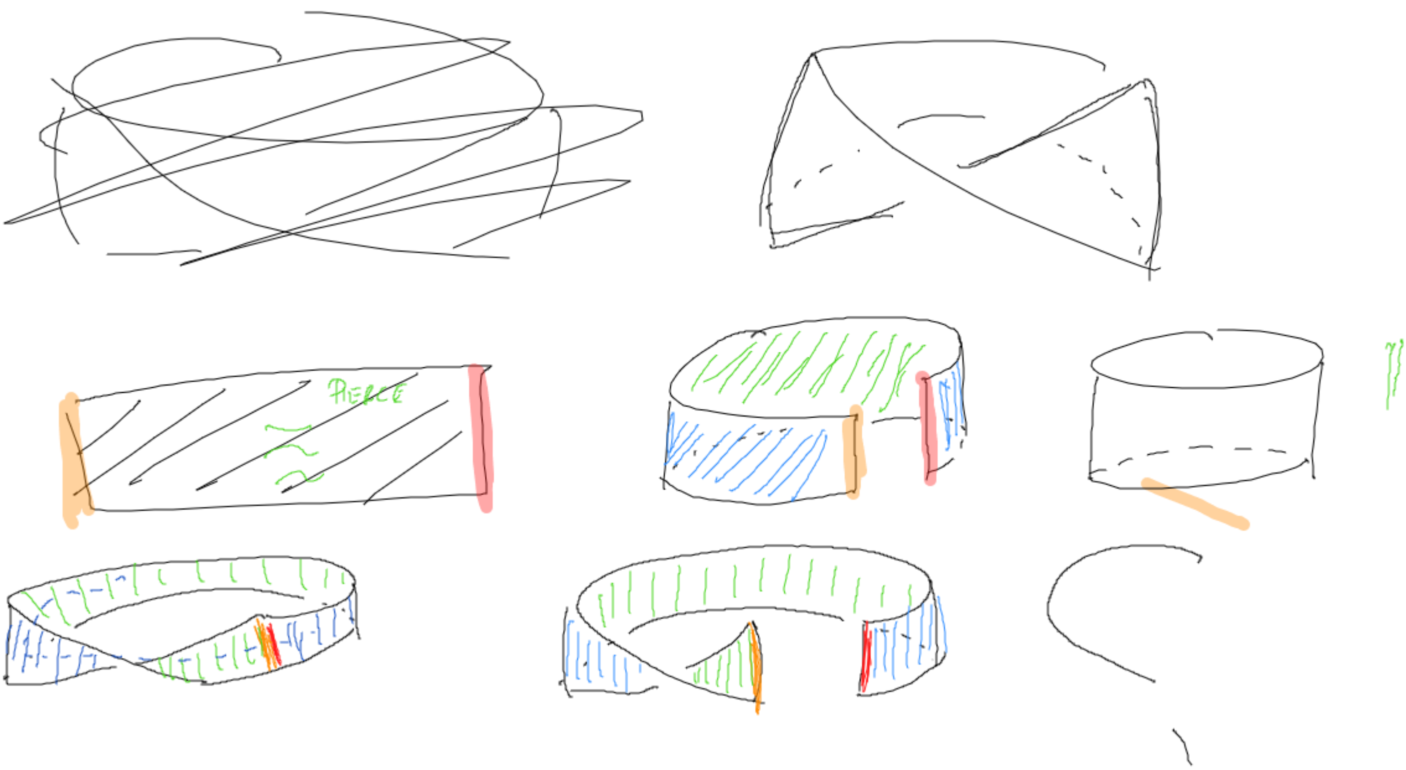
- $\pi_1(\mathbb{R}^n, x_0)$ consists of one element.
- $\pi_1(S^1, x_0)$ is countably infinite.
- $\pi_1(\mathbb{RP}^2, x_0)$ consists of two elements.



<u>Fact</u> Fix a homeomorphism $f: X \underset{\longleftarrow\, f^{-1}}{\longrightarrow} Y$, with $f(x_0)=y_0$.
Then f induces a bijection $\pi_1(X,x_0) \to \pi_1(Y,y_0)$.

$$[\gamma] \longmapsto [f\circ\gamma]$$
$$[f^{-1}\circ\gamma'] \longleftarrow\!\!\!| \; [\gamma']$$


Cannot be shunk to a point.
CAN be shunk.
RP^2
(P^2)
1
RP^2
2
FLIP
FRONT
ACTUAL
3


PIERCE

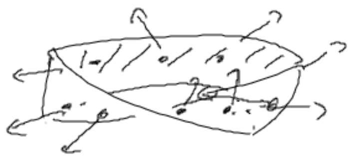
CLAIM If a two-dim. shape

contains a copy of the Möbius band,

it does NOT have "a front and a back."

(It is NOT ORIENTABLE)

(It does NOT admit a $\underbrace{\text{normal vector field}}_{\text{non-vanishing}}$.)



Exer $\mathbb{R}P^2$ contains a Mobius band

Fix a space X, and fix a point $x_0 \in X$.

A <u>closed</u> <u>path</u> <u>based</u> <u>at</u> <u>x_0</u>

is a $\boxed{\text{continuous}}$ fxn

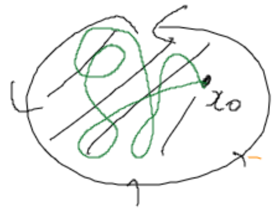
$0 \quad \tfrac{1}{2} \quad \tfrac{2}{3} \quad 1$

$$\gamma: [0,1] \longrightarrow X$$

such that $\gamma(0) = \gamma(1) = x_0$.

$\mathbb{R}^2$

$O(3)$



$\mathbb{RP}^2$

S^1

S^2

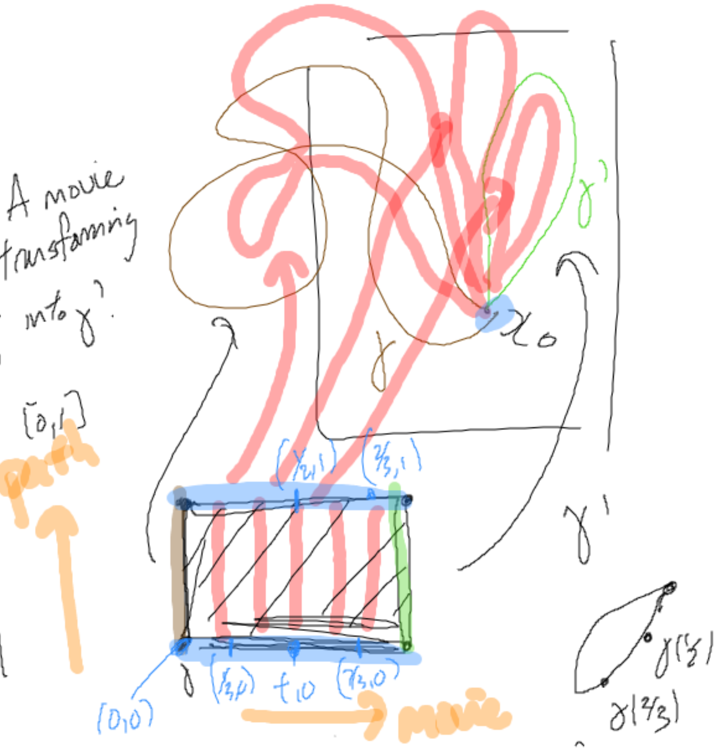
Fix two closed curves based @ x_0, γ and γ'.

<u>Defn</u> A homotopy from γ to γ' is.... a continuous fxn

$$H: \underset{\text{movie time}}{[0,1]} \times \underset{\text{path time}}{[0,1]} \longrightarrow X$$

such that $\forall t$,
- $H(\overset{\text{movie}}{0}, \overset{\text{path}}{t}) = \gamma(\overset{\text{path}}{t})$
- $H(\overset{\text{movie}}{1}, \overset{\text{path}}{t}) = \gamma'(\overset{\text{path}}{t})$, and
- $H(t, \overset{\text{path}}{0}) = H(t, \overset{\text{path}}{1}) = x_0$.

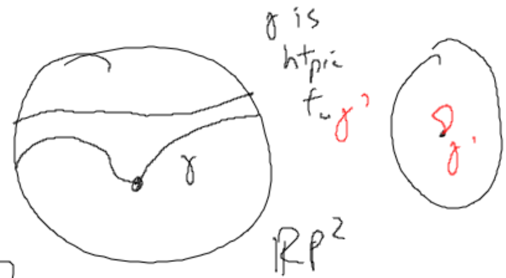
A movie transforming γ into γ'.



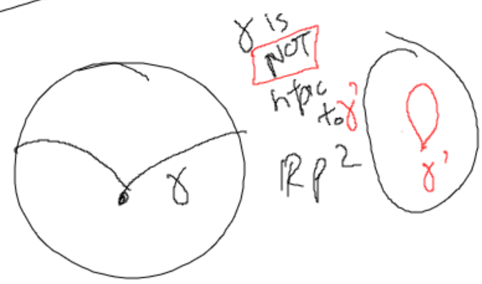
~~Propn~~

Defn We say γ is homotopic to γ' if $\exists$ a homotopy from γ to γ'.

Propn. "Being homotopic" is an equivalence relation.



Two weeks ago



Remark "Homeomorphism" is a(n equivalence) relation of spaces.

But "homotopy" is " " of closed curves based @ x_0.

Defn Fix a space X, and a point $x_0 \in X$.

We define

$$\pi_1(X, x_0) := \{\text{closed curves based @ } x_0\} / \text{homotopy}$$

This is called the fundamental group of X (based @ x_0).