Fix a space X, and a point $x_0 \in X$

Defn

$$\pi_1(X, x_0) := \left\{ \gamma : [0,1] \to X \;\middle|\; \begin{array}{l} \gamma \text{ is cntin.} \\ \gamma(0) = \gamma(1) = x_0 \end{array} \right\} \Big/ \text{htpy rel } x_0$$

The fundamental group of X (based @ x_0)

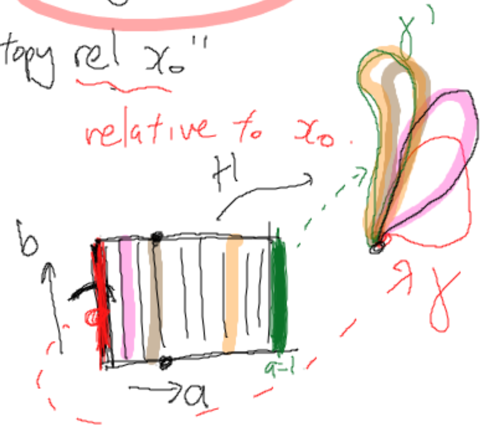
"homotopy rel x_0"

relative to x_0.

(We say "γ is homotopic to γ' rel x_0" if $\gamma \sim \gamma'$

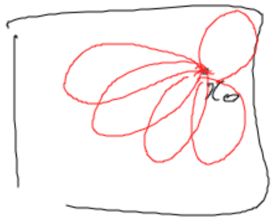
$\exists$ cntns fxn $H : [0,1]_a \times [0,1]_b \longrightarrow X$ s.t.

- $H(0,b) = \gamma(b)$
- $H(1,b) = \gamma'(b)$
- $\forall a \; H(a,0) = H(a,1) = x_0$)



Last time (facts w/o proof)

- $\pi_1(\mathbb{R}^n, x_0) \overset{\text{bijection}}{\cong} *$ ← a one point set.



- $\pi_1(\mathbb{RP}^2, x_0)$ has two elements

- If $\exists$ homeomorphism $\phi: X \to Y$ s.t. $\phi(x_0) = y_0$,
  then $\pi_1(X, x_0) \underset{\text{bijection}}{\cong} \pi_1(Y, y_0)$.

<u>Cor</u> If $\pi_1(X, x_0)$ is not in bijection w/ $\pi_1(Y, y_0)$ then $\nexists$ any homeom. from X to Y sending x_0 to y_0.

<u>Ex</u> $\exists$ no homeom. btwn $\mathbb{R}^n$ and $\mathbb{RP}^2$.

Today: $\pi_1(X, x_0)$ is a group.

Defn A group is the data of

(1) A set G

(2) A function $m: G \times G \longrightarrow G$

$(g_1, g_2) \longmapsto m(g_1, g_2) =: g_1 g_2$

such that

(a) $\exists$ an element $e \in G$ s.t. $\forall g$, $eg = g$ and $ge = g$

(b) $\forall$ elements $g_1, g_2, g_3 \in G$, $m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$

$(g_1 g_2) g_3 = g_1 (g_2 g_3)$

(c) $\forall g \in G$, $\exists h \in G$ s.t. $gh = e$ and $hg = e$.

Motivation: What is a symmetry?

"Something you can do to an object w/o changing some essential thing."

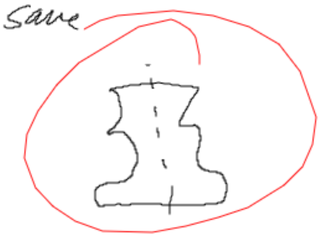
What properties does the collection G of all symmetries (of some object X) have?

- You can do one symmetry after the other. $\forall g \in G, \forall h \in G$, "$hg$" makes sense. ($\exists$ fn $m: G \times G \to G$)
- You can undo a symmetry. $\forall g \in G, \exists h \in G$ s.t. $hg = e$
- "Do nothing" should be a symmetry. e. $he = h$ and $eh = h$.

$g_1 g_2 g_3$

$(g_1 g_2) g_3 = g_1 (g_2 g_3)$

- When folded along some axis, one side is the same as the other.



- A function from a shape to itself?
- $f(a,b) = f(b,a)$

- Take an object, rotate or flip it.

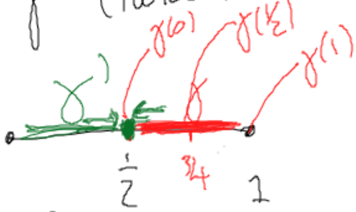
What is the group operation on π_1?

Given two elements $[\gamma], [\gamma'] \in \pi_1(X, x_0)$, how do we obtain a third?

equivalence class of γ.

$[\gamma] \in \pi_1(X, x_0) = \{\gamma \dots\}/\sim$

We can obtain a new loop called "do γ', and then do γ." (twice as fast).



$$\gamma \# \gamma'(t) = \begin{cases} \gamma'(2t) & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Notation

~~Defn~~: The group operation on $\pi_1(X, x_0)$ is:

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$$
$$([\gamma_1], [\gamma_2]) \longmapsto [\gamma_1 \# \gamma_2]$$

Theorem $\pi_1(X, x_0)$ (equipped w/ $\#$)

is a group.

Proof Next time.

Defn A group isomorphism

is a bijection $\phi: G \to G'$

such that $\forall g_1, g_2 \in G$,

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2).$$

Fact. a group $G = (G, m)$

- If G has only one element, we know its group operation uniquely.

  ($G = \{*\}$ $\Rightarrow e = *$ & $* * = *$.)

- If G and G' are both groups w/ two elements, then they are equivalent. ie, $\exists$ a group isom. $G \xrightarrow{\cong} G'$.

If you believe $\pi_1(\mathbb{RP}^2, x_0)$ has two elements,
then you know "what group" it is

Consider the set

$$\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}/2\mathbb{Z} \cong C_2 := \{\overset{\text{odd}}{O}, \overset{\text{even}}{E}\}$$

$\mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}$

$0+0=0 \quad 1+1=0$

$0+1=1+0=1$

Define an operation as follows:

$$C_2 \times C_2 \longrightarrow C_2$$

$$\begin{cases} O\,O = E \\ O\,E = O \\ E\,O = O \\ E\,E = E \end{cases}$$

Ex: $\pi_1(S^1, x_0)$ is countably infinite

$\cong$

$\mathbb{Z}$

$n \longmapsto$

(There are infinitely many such groups.)

A top. space X is ((n)) two-dimensional (manifold)

if : $\forall x \in X$

$\exists$ open $U \subset X$, $x \in U$,

such that U is homeomorphic

to $\mathbb{R}^{2(n)}$.