$$\pi_1(X, x_0) \times \pi_1(Y, y_0) \qquad \pi_1(X \times Y, (x_0, y_0))$$

$$\begin{array}{c} \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ \shortparallel \\ \left\{ \underline{([\gamma], [\gamma'])} \;\middle|\; \begin{array}{l} \gamma : [0,1] \to X \; \cdots \\ \gamma' : [0,1] \to Y \; \cdots \end{array} \right\} \end{array} \qquad \begin{array}{c} \pi_1(X \times Y, (x_0, y_0)) \\ \shortparallel \\ \left\{ \gamma : [0,1] \to X \times Y \;\middle|\; \gamma(0) = \gamma(1) = (x_0, y_0) \right\} \\ \Downarrow \\ \underline{[\gamma]} \end{array}$$

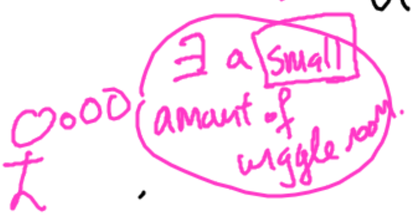
$\sim$

<u>Last time</u>: Dimension.

<u>Q</u>: What's the dimension of $O_n(\mathbb{R})$?

$$O_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T A = I\}.$$

<u>Defn</u> A space $X$ is called <u>$d$-dimensional</u> if $\forall\, x \in X$, $\exists$ an small open subset $U \subset X$ s.t.
- $x \in U$
- $U \cong \mathbb{R}^d$.





## Remark

$$\mathrm{Ball}(x,\varepsilon) \subset \mathbb{R}^d.$$

is homeomorphic to $\mathbb{R}^d$.

To see this, note $\exists$ a homeom.

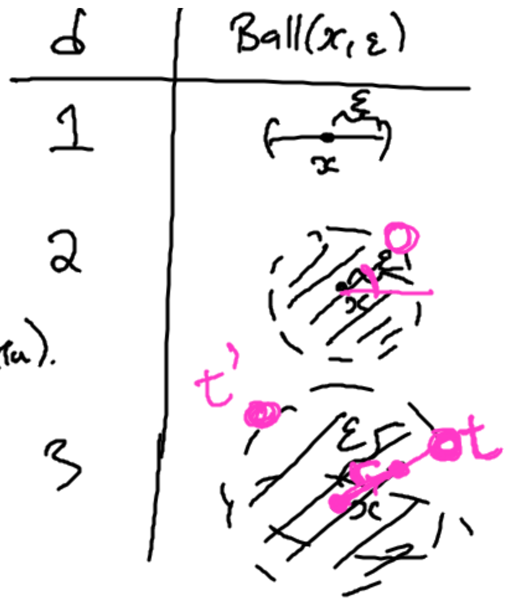
$$[0,\varepsilon) \xrightarrow{\cong} [0,\infty)$$

Exer. (HINT: ArcTan).

And note (using polar coordinates)

$$\mathrm{Ball}(x,\varepsilon) \cong \{(r,t) \mid r\in[0,\varepsilon) \text{ and } t\in S^{d-1}\}/\sim \qquad (0,t)\sim(0,t')$$

$$\cong \{(r,t) \mid r\in[0,\infty) \text{ and } t\in S^{d-1}\}/\sim \;\cong\; \mathbb{R}^d$$



**Upshot** This notion of dimension does NOT depend on whether $X$ is (presented as) some subspace of Euclidean space.

(X is d-dim. if every point admits an open nbhd $\cong$ to $\mathbb{R}^d$)

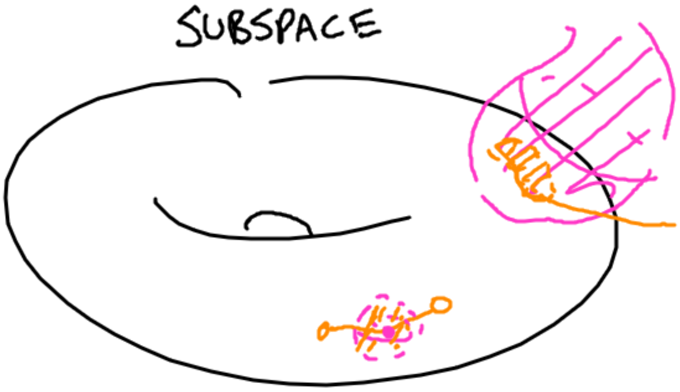
$\boxed{\text{INTRINSIC}}$

$\boxed{\text{EXTRINSIC}}$

$S^1 \subset \mathbb{R}^2$

$$S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$$

SUBSPACE



$\tilde{U} \subset \mathbb{R}^3$

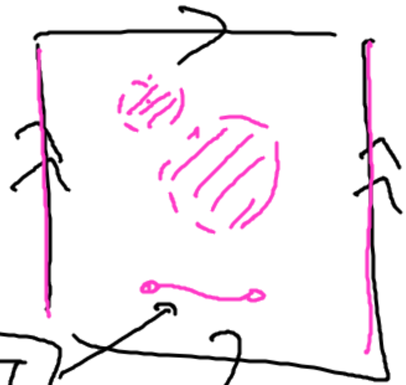
$\cup$

$U \subset X$

$U \times V$

$\cap$

$\underbrace{S^1 \times S^1}_{\text{PRODUCT TOPOLOGY}} \subset \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$



NOT OPEN.

QUOTIENT

$\boxed{[0,1]\times[0,1]/\sim} \supset U$

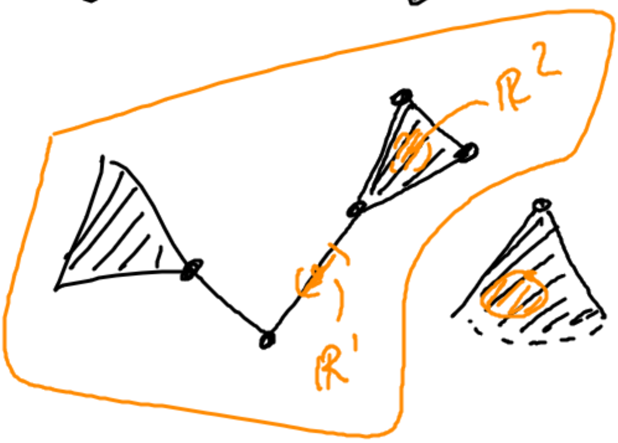
$[0,1]\times[0,1] \supset \tilde{U}$

$\not\cong \mathbb{R}^2$

<u>Pf</u>:

$\forall\, y \in \mathbb{R}^2,$

$\pi_1(\mathbb{R}^2 \setminus \{y\}, x_0) \cong \mathbb{Z}.$





OTOH, choose $x \in$ 


$\pi_1($  $\setminus \{x\}, x_0) \cong *.$

**Prop<u>n</u>** If $X$ is $d$-dim.
and $Y$ is $e$-dimensional,
then $X \times Y$ is $e+d$-dimensional.
~~$\max(e,d)$-dimensional~~.

$S^1 \times S^1$ ↝ 2-dim.

$S^1$ 1-dimensional


**Pf**: $\forall\ (x,y) \in X \times Y$, NTS $\exists$ open nbhd $(x,y) \in U \cong \mathbb{R}^{d+e}$

B/c $\dim X = d$, $\exists$ $U_x \subset X$ s.t. $x \in U_x \cong \mathbb{R}^d$.

" " $Y = e$ .. $U_y \subset Y$ .. $y \in U_y \cong \mathbb{R}^e$.

$U_x \cong \mathbb{R}^d$

By defn of product topology, $U_x \times U_y \subset X \times Y$ is open, and $(x,y) \in U_x \times U_y$.

OTOH $U_x \times U_y \cong \mathbb{R}^d \times \mathbb{R}^e \cong \mathbb{R}^{d+e}$. //

Finding $\boxed{\dim O_n(\mathbb{R})}$ "locally" uses this fact.

$$\boxed{O_n(\mathbb{R})} \ni A = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & & \vdots \\ \vdots & \ddots & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

unit length.

orthonormal basis of $\mathbb{R}^{n-1}$

$\mathbb{R}^{n-1}$

$S^{n-1}$

$$A \simeq \left( \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}, \begin{pmatrix} \ \end{pmatrix} \right)$$

$S^{n-1}$, $O_{n-1}(\mathbb{R})$

ie, an element of $O_{n-1}(\mathbb{R})$

$$\begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix} \in \mathbb{R}^n$$

$\cap$
$S^{n-1}$

Upshot

Near $A$,
$O_n(\mathbb{R}) \supset U \cong S^{n-1} \times O_{n-1}(\mathbb{R})$

# Combinatorial Topology:


| V | E | F | V-E+F |
|---|---|---|---|
| | | | 2 |
| | | | 2 |
| | | | 2 |

2 is called the **Euler characteristic** of $S^2$.

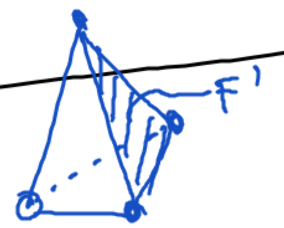
Secret: We've replaced a space ($S^2$) w/ combinatorial data (polyhedra) to compute a # that secretly depends ONLY on homeomorphism type

Defn A simplicial complex ("an abstract simplicial complex")
is the data of
- a set $V$, and
- a subset $S$ of $\mathcal{P}(V)$.

$\mathcal{P}(V)$ = power set of $V$ = $\{A \subset V\}$.


s.t. (0) $\forall v \in V$, $\{v\} \in S$

(1) If $F \in S$ and $F' \subset F$ then $F' \in S$.



Given $(V, S)$ do the following:

$V = \{u, v, w, x\}$

$S = \{ \{u,v\}, \{v,w\}, \{u,w\}, \{x,w\}, \{u,v,w\} \}$

- (0) $\forall\, v \in V$, draw a dot (vertex)
- (1) $\forall\, E \in S$ having exactly two elmts ($E = \{v_0, v_1\}$) draw an edge btwn those 2 elmts
- (2) $\forall\, T \in S$ having exactly 3 elmts ($T = \{v_0, v_1, v_2\}$) draw a ~~T~~RIANGLE
- (3) ... (n) $\forall\, T \in S$ w/ exactly $n+1$ elmts, draw an n-simplex.