(Abstract) Simplicial Complexes

Defn A(n abstract) simplicial complex is the data of

- A set V
- A subset $S \subset P(V)$ of finite subsets of V

s.t. (i) $\forall\, v \in V$, $\{v\} \in S$

(ii) $\forall\, T \in S$, $\forall\, T' \subset T$, $T' \in S$.

$T = \{v_1, \dots, v_n\}$ $\Uparrow$ simplex whose vertices are $v_1, \dots, v_n$

Infomally: An abstract simplicial complex is a set S (of sets) closed under the operation of taking subsets.

Defn A(n abstract) simplicial complex is a set S of finite sets s.t.

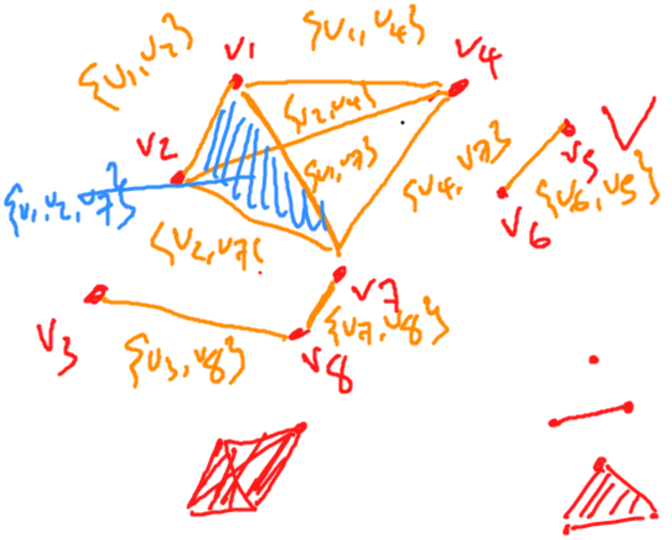
if $\cdot\, T \in S$

$\cdot\, T' \subset T$

then $T' \in S$.

VERY COMBINATORIAL

Given an abstract simplicial cplx, can construct a space made of Δ^0s, Δ^1s, Δ^2s, Δ^3s, etc...



Defn Fix $k \geq 0$. The k-simplex is the subspace

$$\Delta^k := \left\{ (x_0, x_1, \dots, x_k) \;\middle|\; \forall i,\ x_i \geq 0 \text{ and } \sum x_i = 1 \right\} \subset \mathbb{R}^{k+1}$$

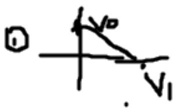
Now you can specify/define
a ton of spaces, just from combinatorial data!

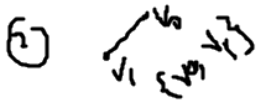
Defn Fix simplicial complexes (V,S) and (V',S').
A simplicial map is a fxn $f: V \longrightarrow V'$
s.t. if $T \in S$ then $f(T) \in S'$.

If $T = \{v_1, \dots, v_n\}$
then $f(T) = \{f(v_1), \dots, f(v_n)\}$

$\boxed{⚠ \ \#f(T) \leq \#T.}$

Exer Draw each of the follwing simplical complexes:

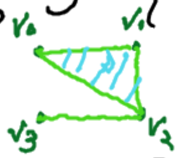




(1) $V=\{v_0,v_1\}$ $\quad S=\{\{v_0\},\{v_1\},\phi\}$

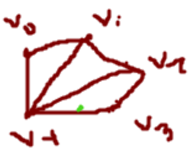
(2) " $\quad S=\{\{v_0\},\{v_1\},\{v_0,v_1\},\phi\}$

(3) $V=\{v_0,v_1,v_2,v_3\}$ $\quad S=\{\{v_0\},\{v_1\},\{v_2\},\{v_3\},\{v_0,v_1\},\{v_1,v_2\},\{v_0,v_2\},$
$\{v_0,v_1,v_2\},\{v_2,v_3\},\phi\}$



$\{v_0,v_1,v_2\}$

(4) $V=\{v_0,v_1,v_2,v_3,v_4\}$ $\quad S=\underbrace{\mathcal{P}(\{v_0,v_1,v_2,v_3\})}\cup\{\{v_4\},\{v_1,v_2,v_4\},\{v_1,v_4\},$
$\{v_2,v_4\}\}$



Q: Can we make shapes like S^2, $\mathbb{RP}^2$, $T^2 = S^1 \times S^1$?

(as a simplical complex?)

$$S^1 \times S^1 = T^2$$

$$\underbrace{S^1 \times \cdots \times S^1}_{k} = T^k$$

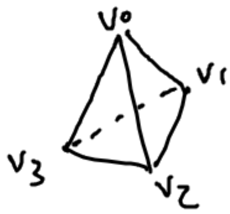
k-dimensional torus

$$S^1 = T^1$$

A: Yes, and non-uniquely so.

Ex Let $V = \{v_0, v_1, v_2, v_3\}$, $S = \mathcal{P}(V) \setminus \{\{v_0, v_1, v_2, v_3\}\}$

The associated space is



(no interior) $\cong S^2$.

$A_0 =$ # vertices $\{v\}$

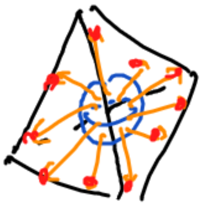
$A_1 =$ # edges $\{v_0, v_1\}$

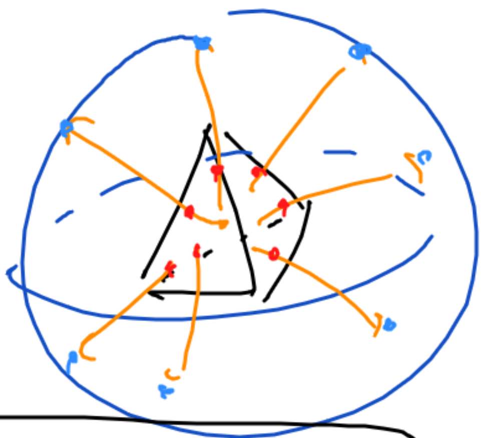
$A_2 =$ # $\triangle$s

$A_3 =$ # 3-simplices

$A_4 =$ # 4-simplices

etc.





Defn Fix a simplicial complex (V, S) ASSUME S IS FINITE

Let A_k be the # of elements of S w/ size exactly $k+1$.

The Euler characteristic

$$\chi(V,S) := \sum_{i=0}^{\infty} (-1)^i A_i \text{ . finite sum}$$

Challenge: Make simplicial complexes that are homeomorphic to $\mathbb{RP}^2$, T^2, T^3.

Ex $A_0 - A_1 + A_2 = V - E + T$

Exer Fix a finite, non-empty set V, and let $S = \mathcal{P}(V)$ (This is a simplex, $(\#V+1)$-dimensional.) Show $\chi(V,S) = 1$.

NOT a simplicial complex!

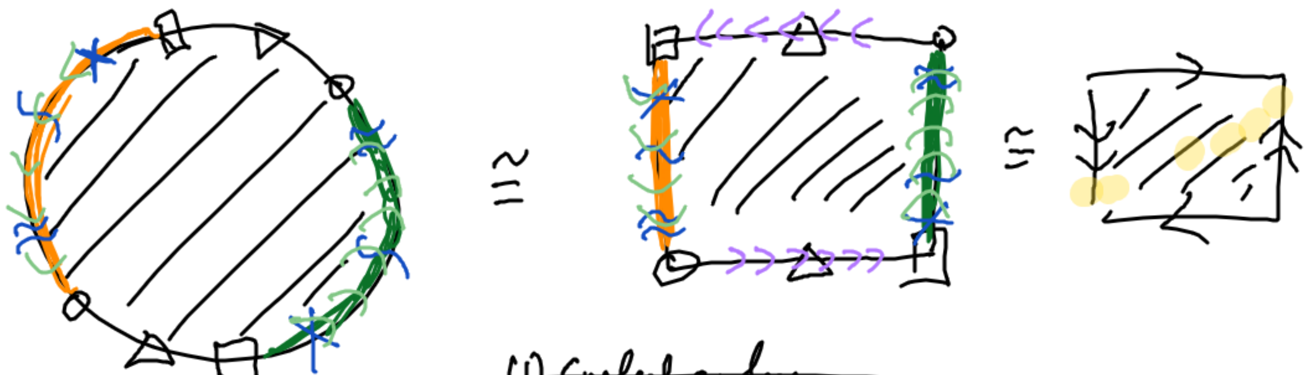
Observation: Simplicial complexes are very clunky.

Why? Given a collection of vertices $\{v_0, \ldots, v_n\} \subset V$, there is at most one simplex spanned by those vertices.

(So the more simplices you want, the more vertices you need.)

$\mathbb{RP}^2$

$$V - E + F = 2 - 2 + 1 = 1 = \chi(\mathbb{RP}^2).$$



Other ways to make spaces

(1) ~~Simplicial complexes~~

(2) CW cplxs

(3) Simplicial sets.

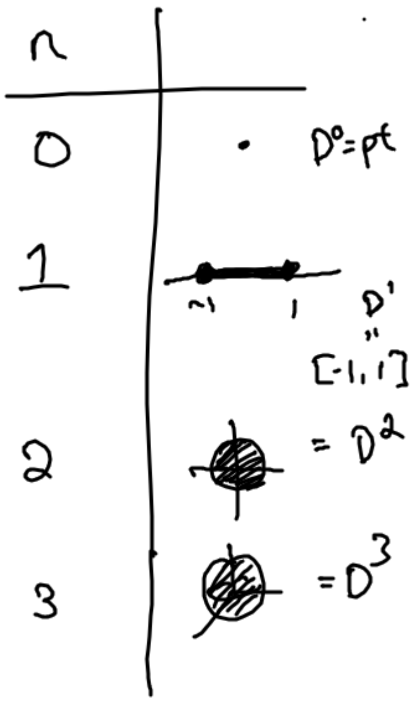
CW complexes: CW cplxes are spaces made out of <u>disks</u>, inductively by dimension.

<u>Recall</u>: $D^n := \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$.

First, choose some set C_0. Let

$$X_0 := \text{"C_0 many D^0s"}$$

$$\cong C_0 \text{ (some cllxn of vertices).}$$

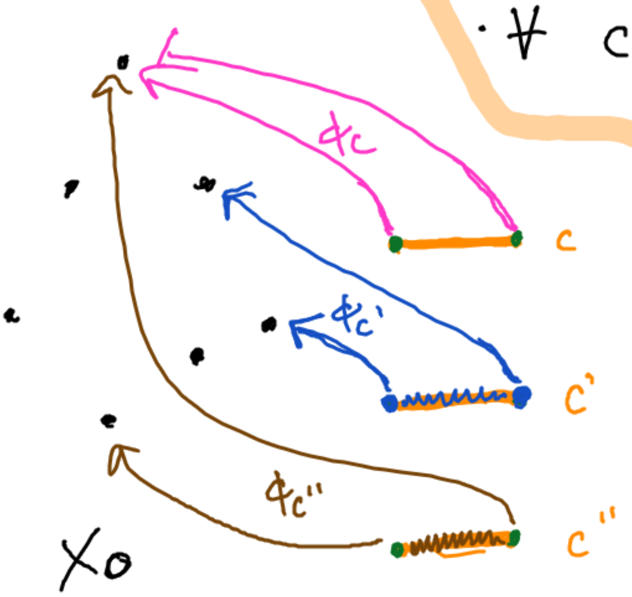


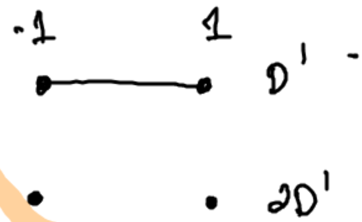
To make X_1, choose

- a set $C_1 = \{c, c', c''\}$

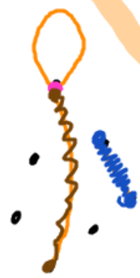
- $\forall\ c \in C_1$, choose a fxn

$\Phi_c : \underbrace{\partial D^1}_{\text{the bdry of } D^1} \longrightarrow X_0$.



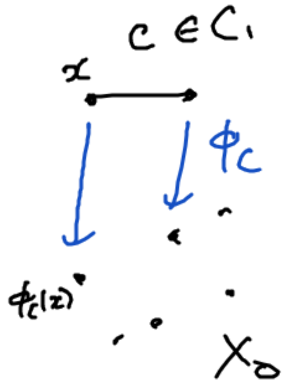


$\rightsquigarrow$ glue orange D^1's to X_0 via Φ_c's.



So X_1 is obtained by gluing

$$C_1 \times D^1 \cong \overbrace{D^1 \cup \cdots \cup D^1}^{\# C_1 \text{ many}}$$



to X_0 along the ϕ_cs.

A fancy way to write X_1:

$$X_1 := \left(X_0 \amalg (C_1 \times D^1) \right) \Big/ \sim$$

$\forall c \in C_1, \forall x \in \partial D^1_c,$
$x \sim \phi_c(x).$

Now fix • a set C_2 $^{=\{E,F,G,H\}}$ (C_2 indexes collection of "2-cells").

• $\forall\, c \in C_2$, a map $\Phi_c : \partial D^2 \longrightarrow X_1$.

Construct X_2 by gluing $C_2 \times D^2 = D^2 \sqcup \cdots \sqcup D^2$ to X_1 along Φs.