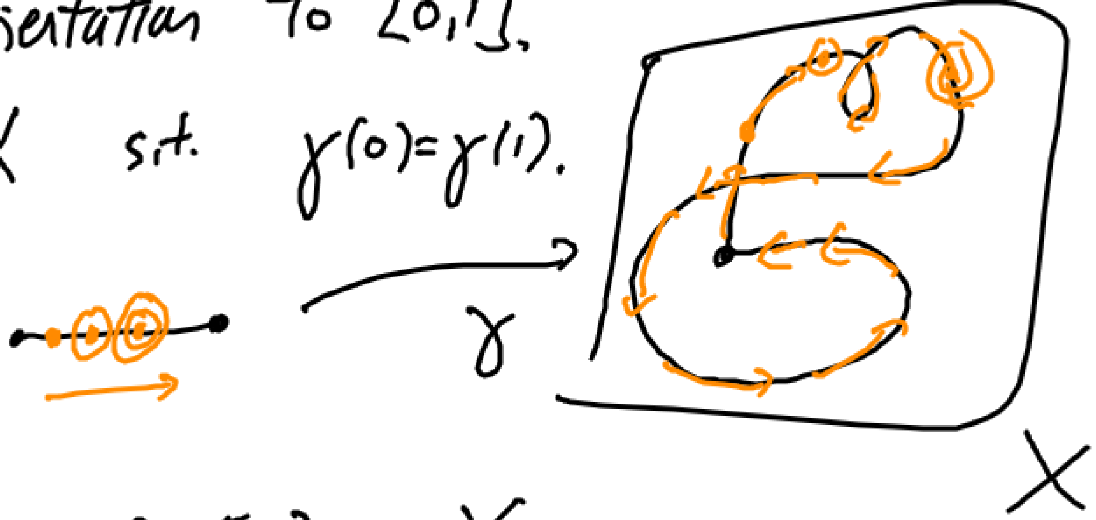
Q (1) Does a loop have a "direction?"

Q (2) Why is it an "abstract" simplicial complex?

(3) How are k-simplices related to simplicial complexes?

A(1) Yes, if you give a direction/orientation to $[0,1]$.

$$\gamma : [0,1] \longrightarrow X \quad \text{s.t.} \quad \gamma(0)=\gamma(1).$$



$$\xrightarrow{\mathbb{R}}$$

A homotopy from γ to γ'?

$$H : [0,1]\times[0,1] \longrightarrow X$$

$$H(0,t) = \gamma(t), \quad H(1,t) = \gamma'(t)$$

Q(2): Why "abstract" in "abstract simp. cplx"?

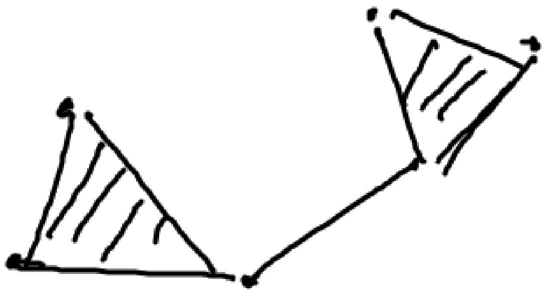
Sets/Combinatorics

"Abstract" simp. cplx

(V, S)

Space

Simplicial Complex.



Q(3): What's the relation btwn "k-simplices" and "simplicial cplxs"?

bones   skeletons

Recall: The k-simplex is the space

eqn. is preserved by permuting coords.

$$\Delta^k := \left\{ (x_0, x_1, \dots, x_k) \;\middle|\; \forall i\ x_i \geq 0,\ \sum_{i=0}^{k} x_i = 1 \right\} \subset \mathbb{R}^{k+1}$$

k	0	1	2	3

Δ^k

One reason for this definition: Faces are defined by setting $x_i = 0$ (for some i).

CW cplxes: (A CW cplx is a space built up inductively out of disks).

on dimension of disks.

(0) Fix a set C_0, and declare $X_0 := C_0$. (A discrete space.)

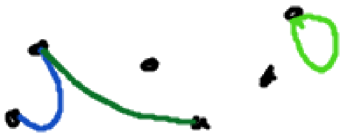
$$= \coprod_{C_0} D^0$$

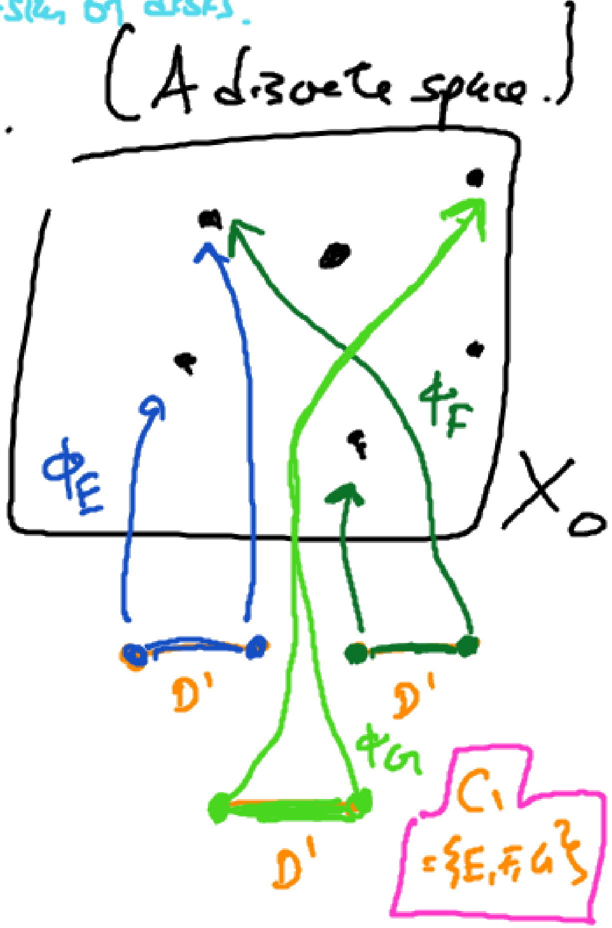
the set of 1-cells.

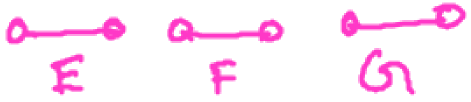
(1) Fix • a set C_1

• $\forall\, c \in C_1$, a fxn $\phi_c : \partial D^1 \to X_0$

Declare $X_1 := \left(X_0 \amalg \left(\coprod_{C_1} D^1 \right) \right) \Big/$ glue along ϕ_cs.





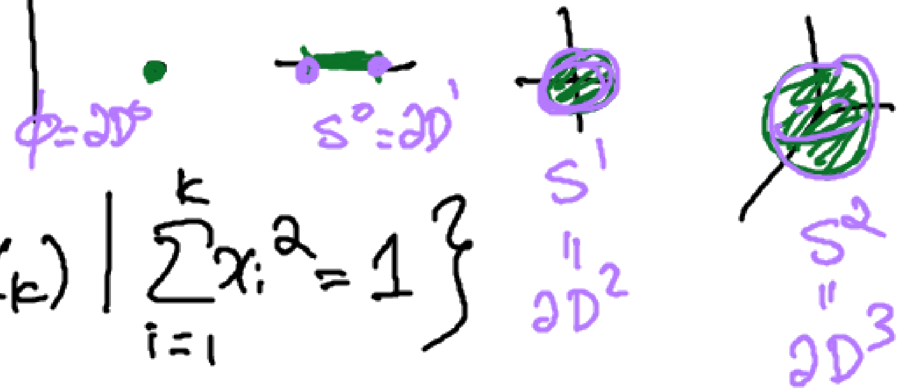


Recall:

$$D^k := \left\{ (x_1, \cdots, x_k) \;\middle|\; \sum_{i=1}^{k} x_i^2 \leq 1 \right\} \subset \mathbb{R}^k$$

the closed k-disk
(aka the k-dimensional disk)

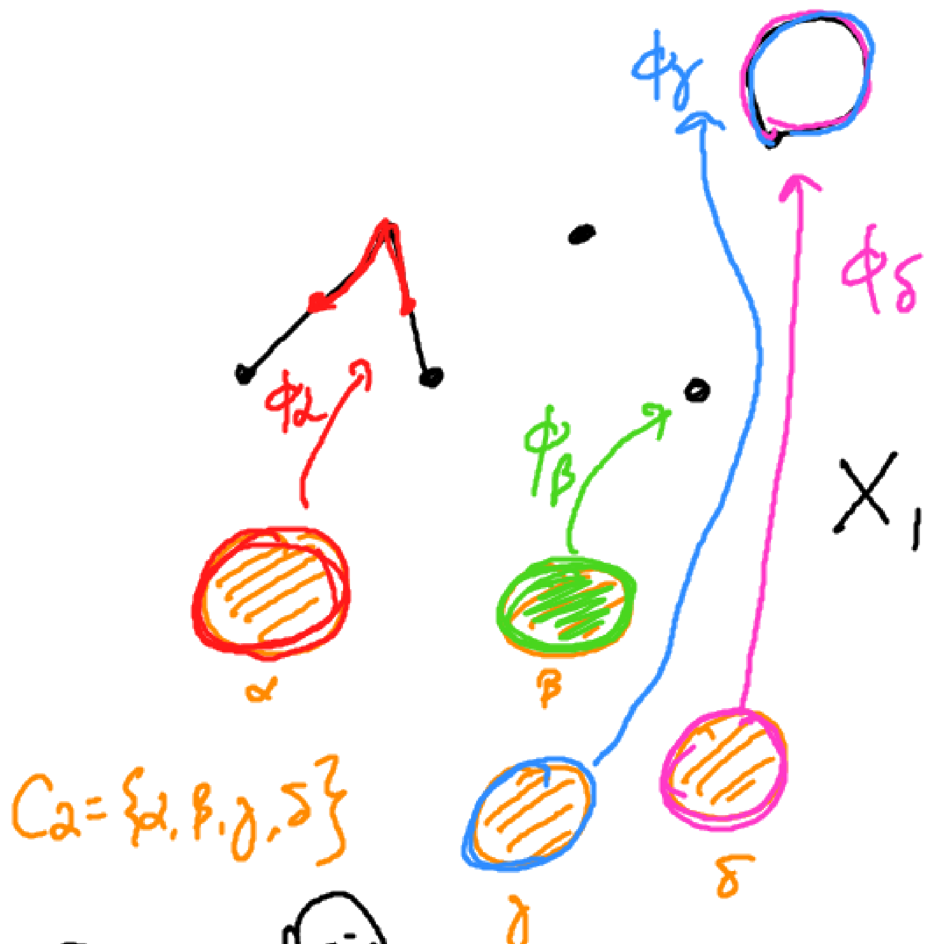
k	0	1	2	3
D^k				



$$\partial D^k := S^{k-1} = \left\{ (x_1, \cdots, x_k) \;\middle|\; \sum_{i=1}^{k} x_i^2 = 1 \right\}$$

(2) Choose

- a set C_2
- $\forall\, c \in C_2$, a fxn $\Phi_c : \partial D^2 \to X_1$



Declare

$$X_2 := \left(X_1 \amalg \left(\coprod_{C_2} D^2 \right) \right) \Big/ \text{glue along } \Phi_c\text{s.}$$

(k) Fix a set C_k

$X_0 \subset X_1 \subset X_2 \subset X_3 \subset X_4 \cdots$

- $\forall\ c \in C_k$, a fxn $\Phi_c : \partial D^k \longrightarrow X_{k-1}$

Declare $X_k := \left(X_{k-1} \amalg \left(\coprod_{C_k} D^k \right) \right) \Big/$ glue along Φ_c s.

$\boxed{X = \bigcup_{k \geq 0} X_k}$ ← X is the CW cplx

Terms: • Each D^k we use is called a <u>k-cell</u>. (So C_k is the set of k-cells.)

k-dimensional cell

- Each Φ_c is called an <u>attaching map</u>.
- X_k is the k-skeleton of X.

Rmk The hardest part about understanding CW cplxs is understanding the continuous maps $\boxed{\phi_c : \partial D^k \to X_{k-1}.}$

nobody knows how to classify all such continuous fxns.

Rmk A CW cplx is determined by

- $C_0, C_1, C_2, \dots$
- $\{\phi_c : \partial D^k \to X_{k-1}\}_{c \in C_k}$

• There's a ϕ_c for every c.

Ex. $C_0 = \{x\}$

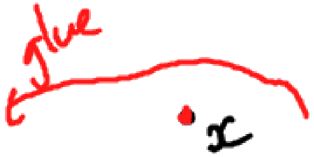
$C_1 = \{e\}.$

 $\phi = C_2 = C_3 = C_4 = \cdots$

$\phi_e : \partial D^1 \to X_0 = \bullet$

X_1

$S^1 \cong$ 



Ex

$C_0 = \{x\}$

$C_1 = \emptyset$

$C_2 = \{d\}$

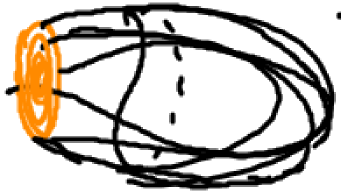
$\emptyset = C_3 = C_4 = C_5 = \cdots$

$\boxed{\text{One 0-cell, one 2-cell}}$

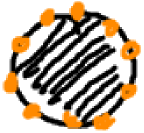
$\phi_d : \underbrace{\partial D^2}_{\cong\, S^1} \longrightarrow X_1 = X_0 = \bullet$

4 vert
4 faces
6 edges
smallest simp cplx $\cong S^2$.

$$X_2 := \left(X_1 \amalg \left(\coprod_{C_2} D^2 \right) \right) \Big/ \text{glue}$$



$$= \left(\bullet \quad \right) \Big/ \text{glue}$$



Ex $C_0 = \{x\}$

$C_1 = \{e\}$ $\qquad X_1 \cong S^1$

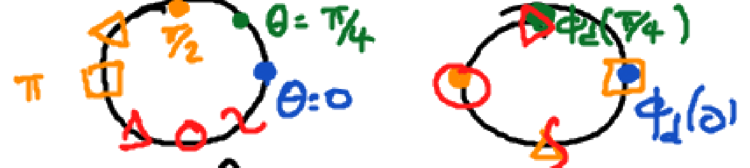
$C_2 = \{d\}$

$C_3 = C_4 = C_5 = \cdots = \emptyset$

$\boxed{\text{What is } X?}$

FOR NEXT TIME.

$C_0, C_1, C_2, \phi_d \rightsquigarrow$ A CW cplx X.



$$\begin{array}{ccccc} \phi_d : & \partial D^2 & \longrightarrow & X_1 \\ & \| & & \cong \\ & S^1 & & S^1 \\ & \| & & \| \\ & \{e^{i\theta}\} & \longrightarrow & \{e^{i\theta}\} \\ & e^{i\theta} & \longmapsto & e^{2i\theta} \end{array}$$

DEFN A CW cplx X is called <u>finite</u> if X is constructed out of finitely many cells.

DEFN Fix a finite CW cplx, X.
The Euler characteristic of X is defined to be

$$\chi(S) = \#C_0 - \#C_1 + \#C_2 - \#C_3 \cdots$$

$$\boxed{\backslash chi} = \sum_{i=0}^{\infty} (-1)^i \, \#C_i$$

(-1)

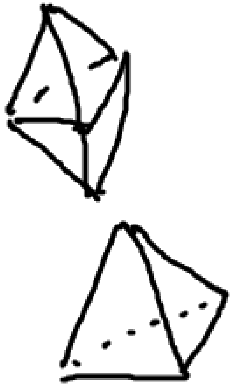
X!??...

$\cdot\ C_0$
$\cdot\ C_1$
$\cdot\ C_2$
$\cdot\ C_3$
C_4
$\vdots$
$\emptyset$
$\emptyset$
$\emptyset$

$\cong$

$W = S^2,$

$\cong$



Thm Fix a space W. Assume W is homeom. to

- A finite CW cplx X, and
- A finite (abstract) simplicial cplx (V, S).

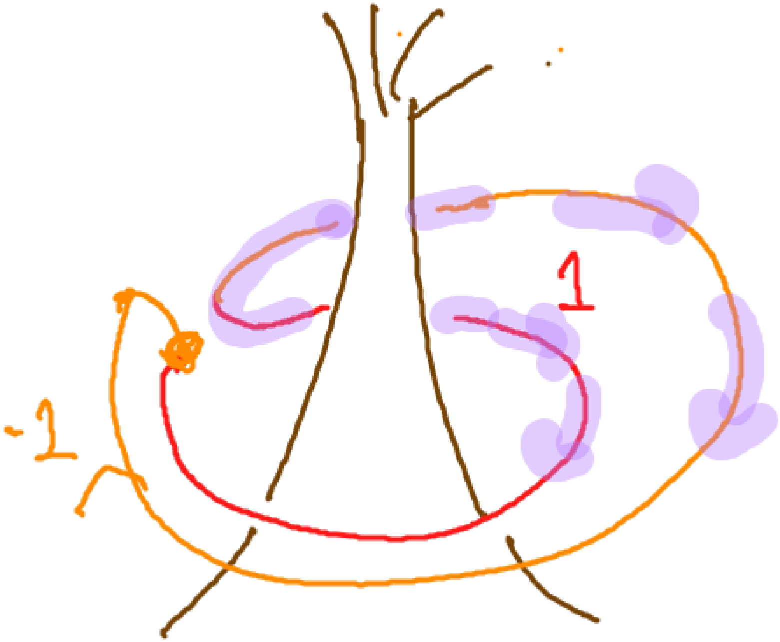
Then

$$\underbrace{\chi(X)}_{\text{Eul. char. of CW}} = \underbrace{\chi(V,S)}_{\text{Eul. ch. of simp cplx}}$$

$$\chi(S^2) = \overset{1}{\#C_0} - \overset{0}{\#C_1} + \overset{1}{\#C_2} = 1 - 0 + 1 = 2.$$

$$\mathrm{pt} \amalg D^n \Big/ \text{glued } \partial D^n \text{ to pt.} \cong S^n$$

$$\boxed{\chi(S^n) = ?}$$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